Proposed Problem

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Problem

Let $n \geq 3$ be an integer, and let $x_1, x_2, \ldots, x_n$ be positive real numbers satisfying

$$\prod_{i=1}^{n} x_i = 1.$$

With the cyclic convention $x_{n+1} = x_1$, prove the sharp inequality

$$\boxed{\sum_{i=1}^{n} \frac{x_i}{x_{i+1}^2} + 3 \sum_{1 \leq i < j \leq n} x_i x_j \geq 3 \sum_{i=1}^{n} x_i + \frac{n(3n-7)}{2}} \tag{1}$$

and determine all equality cases.

Solution

We first record the elementary inequality

$$\frac{1}{t^2} + 2t - 3 = \frac{(t-1)^2(2t+1)}{t^2} \geq 0 \qquad (t > 0). \tag{2}$$

Multiplying (2), with $t = x_{i+1}$, by the positive number x_i gives

$$\frac{x_i}{x_{i+1}^2} + 2x_i x_{i+1} \geq 3x_i. \tag{3}$$

Equality holds in (3) if and only if $x_{i+1} = 1$.

We now separate the cyclic products from all the remaining pairwise products. Let

$$E = \big\{\{1,2\}, \{2,3\}, \ldots, \{n-1,n\}, \{n,1\}\big\}$$

be the edge set of the n-cycle, and let N be the set of unordered pairs $\{i,j\}$ with $1 \leq i < j \leq n$ which do not belong to E. For $n = 3$, every pair is an edge and hence N is empty. Therefore

$$3 \sum_{1 \leq i < j \leq n} x_i x_j - 2 \sum_{i=1}^{n} x_i x_{i+1} = \sum_{\{i,j\} \in E} x_i x_j + 3 \sum_{\{i,j\} \in N} x_i x_j. \tag{4}$$

The right-hand side of (4) is a sum of positive quantities: one copy of x_ix_j for every edge $\{i,j\} \in E$, and three copies of x_ix_j for every non-edge $\{i,j\} \in N$. The total number of these quantities is

$$\begin{aligned} W &= |E| + 3|N| && (1) \\ &= n + 3\left(\binom{n}{2} - n\right) = \frac{n(3n-7)}{2}. && (5) \end{aligned}$$

Each variable x_i occurs in the two products corresponding to the two edges incident with i. It also occurs in $n-3$ non-edge products, each taken three times. Thus its total exponent in the product of all the W quantities is

$$2 + 3(n-3) = 3n - 7.$$

Consequently, using the hypothesis $\prod_{i=1}^n x_i = 1$, the product of those W quantities equals

$$\prod_{i=1}^n x_i^{3n-7} = \left(\prod_{i=1}^n x_i\right)^{3n-7} = 1.$$

The arithmetic–geometric mean inequality applied to them now yields

$$\sum_{\{i,j\}\in E} x_ix_j + 3\sum_{\{i,j\}\in N} x_ix_j \geq W = \frac{n(3n-7)}{2}. \tag{6}$$

Finally, combining (3), summed over $i = 1, \dots, n$, with (4) and (6), we obtain

$$\begin{aligned} &\sum_{i=1}^n \frac{x_i}{x_{i+1}^2} + 3\sum_{1\leq i<j\leq n} x_ix_j \\ &\quad = \sum_{i=1}^n \left(\frac{x_i}{x_{i+1}^2} + 2x_ix_{i+1}\right) + \left(3\sum_{1\leq i<j\leq n} x_ix_j - 2\sum_{i=1}^n x_ix_{i+1}\right) \\ &\quad \geq 3\sum_{i=1}^n x_i + \frac{n(3n-7)}{2}, \end{aligned}$$

which is precisely (1).

If equality holds in (1), equality must hold in every instance of (3). By (2), this forces $x_{i+1} = 1$ for every i, and hence

$$x_1 = x_2 = \cdots = x_n = 1.$$

Conversely, these values clearly give equality in (1). Thus equality holds if and only if

$$\boxed{x_1 = x_2 = \cdots = x_n = 1}.$$

It remains to verify sharpness. Suppose the constant in (1) were replaced by a constant C_n. Substituting $x_1 = x_2 = \cdots = x_n = 1$ would give

$$n + 3\binom{n}{2} \geq 3n + C_n,$$

so necessarily

$$C_n \leq n + 3\binom{n}{2} - 3n = \frac{n(3n-7)}{2}.$$

Since (1) proves the inequality with this value, the constant is best possible.