A Triangle-Altitude Inequality:
Exact Identities, Sharp Refinements, Moment Generalizations, and Applications

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Abstract

For a nondegenerate Euclidean triangle, we study the cyclic altitude expression

$$\mathcal{T} = \sum_{\text{cyc}} \frac{h_b h_c}{bc}.$$

The elementary but decisive observation is that its three summands are $\sin^2 A, \sin^2 B, \sin^2 C$. This gives several exact forms for $\mathcal{T}$, a self-contained proof of the proposed inequality

$$\frac{9r}{2R} \leq \mathcal{T} \leq \frac{9}{4},$$

strictly stronger bounds involving r/R, exact remainder terms, geometric information about the type of the triangle, and generalizations to arbitrary positive powers of the summands. We also derive a recurrence for all integral moments, an exact reciprocal identity, a center-distance decomposition of the lower-bound deficit, and applications to the orthic triangle and to quantitative side dispersion.

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1 Statement and notation

Let ABC be a nondegenerate Euclidean triangle. As usual, its side lengths opposite A, B, C are a, b, c, its corresponding altitudes are h_a, h_b, h_c, its area is Δ, its semiperimeter is s, and its inradius and circumradius are r and R, respectively. We write

$$\mathcal{T} := \sum_{\text{cyc}} \frac{h_b h_c}{bc} = \frac{h_b h_c}{bc} + \frac{h_c h_a}{ca} + \frac{h_a h_b}{ab}.$$

Triangle inequalities involving s, R, r, the side lengths, and the classical centers form a substantial part of Euclidean inequality theory; standard references include [1, 6, 4]. The starting point of the present note is the following problem attributed to Marin Chirciu.

Theorem 1.1 (Marin Chirciu). *For every nondegenerate triangle,*

$$\boxed{\frac{9r}{2R} \le \mathcal{T} \le \frac{9}{4}.}$$

Equality in either inequality holds if and only if the triangle is equilateral.

We will prove a stronger theorem. The original result will then be an immediate corollary.

2 Exact forms of the altitude expression

Proposition 2.1 (The fundamental reduction). *The following identities hold:*

$$\begin{aligned}
\mathcal{T} &= \sin^2 A + \sin^2 B + \sin^2 C, && (1)\\
&= \frac{a^2+b^2+c^2}{4R^2}, && (2)\\
&= 2 + 2\cos A\cos B\cos C, && (3)\\
&= \frac{s^2-r^2-4Rr}{2R^2}. && (4)
\end{aligned}$$

Proof. Since $h_b = 2\Delta/b$ and $h_c = 2\Delta/c$,

$$\frac{h_b h_c}{bc} = \frac{4\Delta^2}{b^2c^2} = \left(\frac{2\Delta}{bc}\right)^2 = \sin^2 A.$$

Cyclic summation proves (1). The extended sine law $a = 2R\sin A$, and its cyclic analogues, then give (2).

For angles whose sum is π,

$$\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C = 1.$$

Subtracting this identity from 3 proves (3).

It remains to obtain the radii form. The standard identity

$$ab + bc + ca = s^2 + r^2 + 4Rr$$

implies

$$a^2 + b^2 + c^2 = (a+b+c)^2 - 2(ab+bc+ca) = 2(s^2 - r^2 - 4Rr).$$

Combining this with (2) gives (4). For completeness, the identity for $ab + bc + ca$ is also derived in the proof of Lemma 3.1 below. □

The first identity explains the entire problem: the expression involving six metric quantities is simply a familiar symmetric function of the three angles.

3 A stronger two-sided inequality

We first prove the only nontrivial metric lemma needed for the lower bound.

Lemma 3.1 (Gerretsen–Schur inequality). *Every nondegenerate triangle satisfies*

$$s^2 \geq 16Rr - 5r^2. \tag{5}$$

Equality holds if and only if the triangle is equilateral.

Proof. This is the lower half of the classical Gerretsen inequality [2, 5]; the following argument is included for completeness. Set

$$x = s - a, \qquad y = s - b, \qquad z = s - c.$$

Then $x, y, z > 0$, $x + y + z = s$, and Heron's formula gives

$$xyz = r^2 s.$$

If $q = xy + yz + zx$, then

$$\begin{aligned} abc &= (y+z)(z+x)(x+y) \\ &= (x+y+z)(xy+yz+zx) - xyz = s(q - r^2). \end{aligned}$$

On the other hand, $abc = 4R\Delta = 4Rrs$, so

$$q = 4Rr + r^2. \tag{6}$$

This calculation also gives

$$ab + bc + ca = x^2 + y^2 + z^2 + 3q = s^2 + q = s^2 + r^2 + 4Rr,$$

which justifies the identity used in Proposition 2.1.

Schur's inequality of degree three says

$$(x+y+z)^3 + 9xyz \geq 4(x+y+z)(xy+yz+zx).$$

Using $x + y + z = s$, $xyz = r^2 s$, and (6), and then dividing by $s > 0$, we obtain

$$s^2 + 9r^2 \geq 4(4Rr + r^2),$$

which is precisely (5). Because x, y, z are positive, equality in Schur's inequality occurs here exactly when $x = y = z$, or equivalently when $a = b = c$. □

Corollary 3.2 (Euler's inequality). *Every nondegenerate triangle satisfies $R \geq 2r$, with equality if and only if the triangle is equilateral.*

Proof. Use the notation from Lemma 3.1. Two applications of the arithmetic–geometric mean inequality give

$$q = xy + yz + zx \geq 3(xyz)^{2/3}, \qquad s = x + y + z \geq 3(xyz)^{1/3}.$$

Therefore $qs \geq 9xyz = 9r^2 s$, so $q \geq 9r^2$. In view of $q = 4Rr + r^2$, this is equivalent to $R \geq 2r$. Equality in both arithmetic–geometric mean inequalities requires $x = y = z$, which is equivalent to $a = b = c$. □

Theorem 3.3 (Refined bounds)**.** *Put*

$$\rho := \frac{r}{R}.$$

Then $0 < \rho \le \frac{1}{2}$*, and*

$$\boxed{6\rho - 3\rho^2 \le \mathcal{T} \le 2 + \frac{2(1+\rho)^3}{27} \le \frac{9}{4}.} \tag{7}$$

The lower and the final upper bounds are attained simultaneously if and only if the triangle is equilateral.

Proof. From (4) and Lemma 3.1,

$$\begin{aligned}\mathcal{T} &= \frac{s^2 - r^2 - 4Rr}{2R^2} \\ &\ge \frac{16Rr - 5r^2 - r^2 - 4Rr}{2R^2} = 6\rho - 3\rho^2.\end{aligned}$$

For the upper estimate, first suppose that the triangle is right or obtuse. Then $\cos A \cos B \cos C \le 0$, after relabelling if necessary, and (3) gives $\mathcal{T} \le 2$. Thus the stated upper estimate is automatic.

If the triangle is acute, all three cosines are positive. The arithmetic–geometric mean inequality and the identities

$$\cos A + \cos B + \cos C = 1 + 4 \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2} = 1 + \frac{r}{R} = 1 + \rho$$

give

$$\cos A \cos B \cos C \le \left(\frac{1+\rho}{3}\right)^3.$$

Equation (3) now gives the middle upper bound in (7).

Finally, Corollary 3.2 yields $\rho \le \frac{1}{2}$, and hence

$$2 + \frac{2(1+\rho)^3}{27} \le 2 + \frac{2(3/2)^3}{27} = \frac{9}{4}.$$

Equality throughout forces equality in Euler's inequality and in the arithmetic–geometric mean step, either of which characterizes the equilateral triangle. □

Proof of Theorem 1.1. The upper bound is already contained in Theorem 3.3. For the lower bound, Corollary 3.2 gives

$$(6\rho - 3\rho^2) - \frac{9}{2}\rho = \frac{3}{2}\rho(1 - 2\rho) \ge 0.$$

Consequently $\mathcal{T} \ge 6\rho - 3\rho^2 \ge 9\rho/2$, as required. The equality statement follows from Theorem 3.3. □

4 Exact remainder terms and stability

The preceding lower refinement can be written as an exact identity with a nonnegative remainder.

Proposition 4.1 (Gerretsen remainder). *Define*

$$\Gamma := s^2 - 16Rr + 5r^2.$$

Then $\Gamma \geq 0$ *and*

$$\mathcal{T} = 6\rho - 3\rho^2 + \frac{\Gamma}{2R^2}, \tag{8}$$

$$\mathcal{T} - \frac{9}{2}\rho = \frac{3}{2}\rho(1 - 2\rho) + \frac{\Gamma}{2R^2}. \tag{9}$$

Thus the gap in the original lower bound contains both an Euler deficit, measured by $1 - 2r/R$, *and a Schur deficit, measured by* Γ.

Proof. Lemma 3.1 gives $\Gamma \geq 0$. Substitute $s^2 = 16Rr - 5r^2 + \Gamma$ into (4) to obtain (8); subtracting $9\rho/2$ gives (9). □

The remainder has a direct geometric meaning. Let O, I, G denote, respectively, the circumcenter, incenter, and centroid.

Proposition 4.2 (Center-distance decomposition). *One has*

$$\Gamma = 9IG^2, \tag{10}$$

$$\mathcal{T} = 6\rho - 3\rho^2 + \frac{9}{2}\frac{IG^2}{R^2}, \tag{11}$$

$$\mathcal{T} - \frac{9}{2}\rho = \frac{3}{2}\rho\frac{OI^2}{R^2} + \frac{9}{2}\frac{IG^2}{R^2}. \tag{12}$$

Thus equality in the original lower bound is equivalent to the simultaneous coincidence $O = I = G$.

Proof. The classical Gerretsen identity is

$$9IG^2 = s^2 - 16Rr + 5r^2.$$

For a short verification, regard the vertices as position vectors and write

$$I = \frac{aA + bB + cC}{2s}, \qquad G = \frac{A + B + C}{3}.$$

If $\alpha + \beta + \gamma = 0$, then

$$\|\alpha A + \beta B + \gamma C\|^2 = -\alpha\beta c^2 - \beta\gamma a^2 - \gamma\alpha b^2.$$

Substitution of $\alpha = a/(2s) - 1/3$, and its cyclic analogues, followed by $abc = 4Rrs$ and $ab + bc + ca = s^2 + r^2 + 4Rr$, yields (10). Equation (11) now follows from (8). Finally, Euler's center formula $OI^2 = R(R - 2r)$ converts (9) into (12). □

There is a second, complementary remainder formula. It records exactly how much is lost in the Cauchy–Schwarz approach to the original problem.

Proposition 4.3 (Cauchy remainder)**.** *Let*

$$V := (a-b)^2 + (b-c)^2 + (c-a)^2.$$

Then

$$\boxed{\mathcal{T} = \frac{s^2}{3R^2} + \frac{V}{12R^2}.} \tag{13}$$

In particular,

$$\mathcal{T} \geq \frac{9r}{2R} + \frac{V}{12R^2}. \tag{14}$$

Proof. The elementary variance identity

$$u^2 + v^2 + w^2 = \frac{(u+v+w)^2}{3} + \frac{(u-v)^2 + (v-w)^2 + (w-u)^2}{3}$$

is applied to

$$u = \frac{1}{bc}, \qquad v = \frac{1}{ca}, \qquad w = \frac{1}{ab}.$$

Since $u + v + w = 2s/(abc)$, $abc = 4R\Delta$, and

$$(u-v)^2 + (v-w)^2 + (w-u)^2 = \frac{V}{a^2b^2c^2},$$

multiplication by $4\Delta^2$ gives (13).

It remains to estimate the first term. By Lemma 3.1 and Corollary 3.2,

$$s^2 \geq 16Rr - 5r^2 \geq 16Rr - \frac{5}{2}Rr = \frac{27}{2}Rr.$$

Substitution in (13) proves (14). □

Formula (14) is a genuine strengthening of the original result: it supplies an explicit positive correction whenever the side lengths are not all equal.

5 Geometric interpretation

The same quantity detects whether the triangle is acute, right, or obtuse.

Corollary 5.1 (Angle-type criterion)**.** *For a nondegenerate triangle,*

type of ABC	*value of* $\mathcal{T}$
acute	$\mathcal{T} > 2$,
right	$\mathcal{T} = 2$,
obtuse	$\mathcal{T} < 2$.

Proof. By (3), the sign of $\mathcal{T} - 2$ is the sign of $\cos A \cos B \cos C$. All three cosines are positive in an acute triangle; exactly one is zero in a right triangle; and exactly one is negative in an obtuse triangle. □

Thus the universal upper bound $9/4$ is attained at the equilateral triangle, while the value 2 is the exact dividing line between acute and nonacute triangles.

5.1 The orthic triangle

Suppose now that ABC is acute, and let D, E, F be the feet of the altitudes from A, B, C, respectively. The triangle DEF is the orthic triangle of ABC.

Proposition 5.2 (Orthic-area formula)**.** *For every acute triangle,*

$$\boxed{\frac{[DEF]}{\Delta} = \mathcal{T} - 2.} \tag{15}$$

Consequently,

$$0 < \frac{[DEF]}{\Delta} \le \frac{2(1+\rho)^3}{27} \le \frac{1}{4}. \tag{16}$$

Equality in the final inequality holds if and only if ABC is equilateral.

Proof. The sides of the orthic triangle are $EF = a\cos A$, $FD = b\cos B$, and $DE = c\cos C$, and its circumcircle is the nine-point circle of radius $R/2$; see, for example, [4, Chapter IX]. Hence

$$[DEF] = \frac{abc\cos A\cos B\cos C}{4(R/2)} = 2\Delta\cos A\cos B\cos C.$$

Formula (15) follows from (3), and (16) follows from Theorem 3.3. □

Thus $\mathcal{T} - 2$ is not merely a sign test for acuteness: in the acute case it is exactly the proportion of the original area occupied by the orthic triangle.

5.2 A side-dispersion interpretation

Let

$$\bar{a} := \frac{a+b+c}{3} = \frac{2s}{3}$$

and define the squared coefficient of variation of the three side lengths by

$$\mathrm{CV}^2 := \frac{(a-\bar{a})^2 + (b-\bar{a})^2 + (c-\bar{a})^2}{3\bar{a}^2}.$$

Proposition 5.3 (Normalized side dispersion)**.** *The altitude expression satisfies*

$$\boxed{\mathcal{T} = \frac{s^2}{3R^2}\left(1 + \mathrm{CV}^2\right).} \tag{17}$$

Equivalently,

$$\mathrm{CV}^2 = \frac{3R^2\mathcal{T}}{s^2} - 1.$$

Thus, once s/R is fixed, $\mathcal{T}$ measures exactly the quadratic departure of the side triple from equality.

Proof. The identity

$$(a-b)^2 + (b-c)^2 + (c-a)^2 = 3\big((a-\bar{a})^2 + (b-\bar{a})^2 + (c-\bar{a})^2\big)$$

gives $\mathrm{CV}^2 = V/(4s^2)$. Substitution in (13) proves (17). □

6 Power sums and moment generalizations

The termwise identity from Proposition 2.1 extends the problem to a full family of moments. Since the triangle is nondegenerate, all three summands are positive, so for every $p \in \mathbb{R}$ we may define

$$\mathcal{T}_p := \sum_{\text{cyc}} \left(\frac{h_b h_c}{bc} \right)^p .$$

Then

$$\mathcal{T}_p = \sin^{2p} A + \sin^{2p} B + \sin^{2p} C. \tag{18}$$

Theorem 6.1 (Positive real powers)**.** *If $p \geq 1$, then*

$$\boxed{3^{1-p}(6\rho - 3\rho^2)^p \leq \mathcal{T}_p \leq \mathcal{T} \leq 2 + \frac{2(1+\rho)^3}{27}.} \tag{19}$$

If $0 < p \leq 1$, then

$$\boxed{6\rho - 3\rho^2 \leq \mathcal{T} \leq \mathcal{T}_p \leq 3^{1-p} \left(2 + \frac{2(1+\rho)^3}{27} \right)^p .} \tag{20}$$

The first bound in (19) *and the final bound in* (20) *are sharp at the equilateral triangle.*

Proof. Put $u_1 = \sin^2 A$, $u_2 = \sin^2 B$, and $u_3 = \sin^2 C$. For $p \geq 1$, the power-mean inequality gives

$$\sum_{i=1}^{3} u_i^p \geq 3^{1-p} \left(\sum_{i=1}^{3} u_i \right)^p ,$$

while $u_i^p \leq u_i$ because $0 < u_i \leq 1$. This proves (19) after applying Theorem 3.3. If $0 < p \leq 1$, then $u_i^p \geq u_i$, and concavity of t^p gives

$$\sum_{i=1}^{3} u_i^p \leq 3^{1-p} \left(\sum_{i=1}^{3} u_i \right)^p .$$

The refined bounds for $\mathcal{T}$ now yield (20). These power-mean facts may also be viewed as standard consequences of Hölder's inequality [3]. ☐

Proposition 6.2 (Log-convexity and interpolation)**.** *For $p, q \in \mathbb{R}$ and $0 \leq \theta \leq 1$,*

$$\mathcal{T}_{\theta p + (1-\theta) q} \leq \mathcal{T}_p^{\theta} \mathcal{T}_q^{1-\theta}. \tag{21}$$

In particular, if $0 < p < q$, then

$$\left(\frac{\mathcal{T}_p}{3} \right)^{1/p} \leq \left(\frac{\mathcal{T}_q}{3} \right)^{1/q} . \tag{22}$$

Proof. Apply Hölder's inequality to $u_i^{\theta p} u_i^{(1-\theta) q}$, where $u_i \in \{\sin^2 A, \sin^2 B, \sin^2 C\}$. This gives (21). To obtain (22), use the endpoints q and 0, take $\theta = p/q$, and note that $\mathcal{T}_0 = 3$. ☐

For integral exponents, the whole family is determined by three geometric invariants.

Theorem 6.3 (Exact recurrence for integral moments)**.** *Define*

$$
\begin{aligned}
E_1 &:= \frac{s^2 - r^2 - 4Rr}{2R^2},\\
E_2 &:= \frac{(s^2 + r^2 + 4Rr)^2 - 16Rrs^2}{16R^4},\\
E_3 &:= \frac{r^2 s^2}{4R^4}.
\end{aligned}
$$

Then

$$\mathcal{T}_0 = 3, \qquad \mathcal{T}_1 = E_1, \qquad \mathcal{T}_2 = E_1^2 - 2E_2, \tag{23}$$

and, for every integer $n \geq 3$,

$$\boxed{\mathcal{T}_n = E_1 \mathcal{T}_{n-1} - E_2 \mathcal{T}_{n-2} + E_3 \mathcal{T}_{n-3}.} \tag{24}$$

Proof. Let $u_1 = \sin^2 A$, $u_2 = \sin^2 B$, and $u_3 = \sin^2 C$. Their elementary symmetric functions are

$$u_1 + u_2 + u_3 = E_1,$$

by Proposition 2.1, and

$$
\begin{aligned}
u_1u_2 + u_2u_3 + u_3u_1 &= \frac{a^2b^2 + b^2c^2 + c^2a^2}{16R^4} = E_2,\\
u_1u_2u_3 &= \left(\frac{abc}{8R^3}\right)^2 = E_3.
\end{aligned}
$$

Indeed,

$$a^2b^2 + b^2c^2 + c^2a^2 = (ab + bc + ca)^2 - 2abc(a + b + c),$$

and the identities $ab + bc + ca = s^2 + r^2 + 4Rr$, $abc = 4Rrs$, and $a + b + c = 2s$ give the displayed formula for E_2. Newton's identities now yield (23)–(24). □

The first negative moment gives an exact companion to the original altitude sum.

Corollary 6.4 (Reciprocal altitude-product identity)**.** *One has*

$$\boxed{\sum_{\text{cyc}} \frac{bc}{h_b h_c} = \sum_{\text{cyc}} \csc^2 A = \frac{(s^2 + r^2 + 4Rr)^2 - 16Rrs^2}{4r^2s^2}.} \tag{25}$$

Proof. The left side is $\mathcal{T}_{-1}$ by (18). Therefore

$$\mathcal{T}_{-1} = \frac{u_1u_2 + u_2u_3 + u_3u_1}{u_1u_2u_3} = \frac{E_2}{E_3},$$

which is (25). □

More generally, for any function $F : (0, 1] \to \mathbb{R}$, one has the exact transfer principle

$$\sum_{\text{cyc}} F\left(\frac{h_b h_c}{bc}\right) = F(\sin^2 A) + F(\sin^2 B) + F(\sin^2 C).$$

Consequently, estimates based on convexity, concavity, monotonicity, or interpolation for the three squared sines translate directly into altitude inequalities.

7 Sharpness and examples

Proposition 7.1 (Best constants in the original form)**.** *The coefficient* $9/2$ *in the lower bound* $\mathcal{T} \geq C(r/R)$ *cannot be increased, and the upper constant* $9/4$ *cannot be decreased.*

Proof. For an equilateral triangle,

$$\rho = \frac{1}{2}, \qquad \mathcal{T} = 3\sin^2\frac{\pi}{3} = \frac{9}{4}.$$

Thus a universal inequality $\mathcal{T} \geq C\rho$ forces $9/4 \geq C/2$, or $C \leq 9/2$, and any universal constant upper bound must be at least $9/4$. □

There is no positive lower bound for $\mathcal{T}$ that is independent of the shape of the triangle. For instance, as triangles with side lengths $1, 1, 2-\varepsilon$ approach degeneracy, their angles approach $0, 0, \pi$, and hence $\mathcal{T} = \sum \sin^2 A \to 0$. This explains why a scale-free shape parameter such as r/R is essential in the lower bound.

Example 7.2 (The 3-4-5 triangle)**.** Here $R = 5/2$, $r = 1$, and $\rho = 2/5$. Since the triangle is right, Corollary 5.1 gives $\mathcal{T} = 2$. The refined estimates read

$$6\rho - 3\rho^2 = \frac{48}{25} = 1.92 \leq 2 \leq 2 + \frac{2(7/5)^3}{27} \approx 2.2033,$$

whereas the original lower bound is only $9\rho/2 = 9/5 = 1.8$. The reciprocal identity gives, in the same example,

$$\sum_{\text{cyc}} \frac{bc}{h_b h_c} = \frac{25}{9} + \frac{25}{16} + 1 = \frac{769}{144}.$$

8 Conclusion

The central identity

$$\frac{h_b h_c}{bc} = \sin^2 A$$

reduces the proposed altitude inequality to a transparent triangle invariant. It yields a shorter proof, the strengthened bounds

$$6\frac{r}{R} - 3\left(\frac{r}{R}\right)^2 \leq \mathcal{T} \leq 2 + \frac{2}{27}\left(1 + \frac{r}{R}\right)^3,$$

exact stability formulas with center-distance interpretations, an angle-type criterion, and a family of positive and negative moment extensions. In particular, the integral moments satisfy a third-order recurrence determined entirely by s, R, r. The same invariant gives the relative area of the orthic triangle in the acute case and a normalized quadratic measure of side dispersion. The constants in the original inequality are best possible because both are forced by the equilateral triangle.

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