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SOLUTIONS

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PROBLEMS FOR JUNIORS

JP.616 Determine the largest positive value of the constant k such that:

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} - 4 \geq k(a + b + c + d - 4)$$

for any positive real numbers a, b, c, d satisfying $ab + bc + cd + da = 4$

Proposed by Vasile Cîrtoaje – Romania

Solution by proposer

For $a = c = x > 0$ and $b = d = \frac{1}{x}$, the constraint is satisfied and the inequality becomes

$$x^2 + \frac{1}{x^2} - 2 \geq k\left(x + \frac{1}{x} - 2\right), \quad (x-1)^2[(x-1)^2 + 4 - k] \geq 0.$$

For $x \rightarrow 1$, we get the necessary condition $k \leq 4$. We claim that 4 is the largest positive value of k . To show this, we need to prove that the inequality holds for $k = 4$, that is

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} + 12 \geq 4(a + b + c + d).$$

Let

$$x = \frac{a+c}{2}, \quad y = \frac{b+d}{2}, \quad xy = 1.$$

By the AM-GM inequality, we have

$$\frac{1}{a^2} + \frac{1}{c^2} \geq \frac{2}{ac} \geq \frac{8}{(a+c)^2} = \frac{2}{x^2}.$$

Similarly,

$$\frac{1}{b^2} + \frac{1}{d^2} \geq \frac{2}{y^2}.$$

So, it suffices to show that

$$\begin{aligned} \frac{1}{x^2} + \frac{1}{y^2} + 6 &\geq 4(x+y), & \frac{1}{x^2} + x^2 + 6 &\geq 4\left(x + \frac{1}{x}\right), \\ \left(x + \frac{1}{x}\right)^2 + 4 &\geq 4\left(x + \frac{1}{x}\right), & \left(x + \frac{1}{x} - 2\right)^2 &\geq 0. \end{aligned}$$

If $k = 4$, then the equality occurs when $a = b = c = d = 1$.

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JP.617 Let a, b, c, d be nonnegative real numbers such that

$a \geq b \geq c \geq d$ and $(a + d)(b + c) = 4$. Prove that:

$$\frac{1}{a+3} + \frac{1}{b+3} + \frac{1}{c+3} + \frac{1}{d+3} \geq 1.$$

Proposed by Vasile Cîrtoaje – Romania

Solution by proposer

Denoting

$$x = \frac{a+d}{2}, \quad y = \frac{b+c}{2},$$

we have $xy = 1$. By the AM-HM inequality, we get

$$\frac{1}{b+3} + \frac{1}{c+3} \geq \frac{2}{y+3}.$$

So, it suffices to show that

$$\frac{1}{a+3} + \frac{1}{d+3} + \frac{2}{y+3} \geq 1,$$

i.e.

$$\frac{1}{a+3} + \frac{1}{d+3} \geq \frac{y+1}{y+3}, \quad \frac{2(x+3)}{ad+6x+9} \geq \frac{y+1}{y+3}.$$

From $(y-a)(y-d) \leq 0$, we get $ad \leq 2xy - y^2 = 2 - y^2$, hence

$$\frac{2(x+3)}{ad+6x+9} \geq \frac{2(x+3)}{(2-y^2)+6x+9} = \frac{2(3y+1)}{-y^3+11y+6}$$

So, it suffices to show that

$$\frac{2(3y+1)}{-y^3+11y+6} \geq \frac{y+1}{y+3},$$

which is equivalent to

$$y(y^3 + y^2 - 5y + 3) \geq 0, \quad y(y+3)(y-1)^2 \geq 0.$$

The equality occurs for $a = b = c = d = 1$.

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JP.618 Let $x_1, x_2, \dots, x_m > 0$, where $m \geq 2$, and use the cyclic convention $x_{m+1} = x_1$. If $\alpha > 1$ and $\lambda \geq 1$, prove that

$$\sum_{k=1}^m \frac{x_k((\alpha - 1)x_k + 2x_{k+1})}{\alpha x_k^2 + x_{k+1}^2} + \lambda \sum_{k=1}^m \frac{(x_k - x_{k+1})^2}{x_k^2 + x_{k+1}^2} \geq m,$$

and determine all equality cases.

Proposed by Mihály Bencze – Romania

Solution by proposer

The inequality follows from a termwise estimate. Fix an index k and write

$$x = x_k, \quad y = x_{k+1}.$$

Since $x, y > 0$, all denominators below are positive. First observe the following exact identity:

$$\frac{x((\alpha - 1)x + 2y)}{\alpha x^2 + y^2} + \frac{(x - y)^2}{x^2 + y^2} - 1 = \frac{(\alpha - 1)x^2(x - y)^2}{(\alpha x^2 + y^2)(x^2 + y^2)}.$$

Indeed, after putting the left-hand side over the common denominator

$(\alpha x^2 + y^2)(x^2 + y^2)$, its numerator is

$$\begin{aligned} & x((\alpha - 1)x + 2y)(x^2 + y^2) + (x - y)^2(\alpha x^2 + y^2) \\ & - (\alpha x^2 + y^2)(x^2 + y^2) = (\alpha - 1)x^2(x - y)^2. \end{aligned}$$

We now restore the coefficient λ . Splitting $\lambda = 1 + (\lambda - 1)$ in the second fraction give

$$\begin{aligned} \frac{x((\alpha - 1)x + 2y)}{\alpha x^2 + y^2} + \lambda \frac{(x - y)^2}{x^2 + y^2} - 1 &= \frac{(\alpha - 1)x^2(x - y)^2}{(\alpha x^2 + y^2)(x^2 + y^2)} + (\alpha - 1) \frac{(x - y)^2}{x^2 + y^2} = \\ &= \frac{(x - y)^2}{x^2 + y^2} \left(\lambda - 1 + \frac{(\alpha - 1)x^2}{\alpha x^2 + y^2} \right). \end{aligned}$$

The last expression is nonnegative because $\lambda - 1 \geq 0$, $\alpha - 1 > 0$, and $x, y > 0$.

Consequently, for every $k = 1, \dots, m$

$$\frac{x_k((\alpha - 1)x_k + 2x_{k+1})}{\alpha x_k^2 + x_{k+1}^2} + \lambda \frac{(x_k - x_{k+1})^2}{x_k^2 + x_{k+1}^2} \geq 1.$$

Adding these m inequalities proves the required lower bound m .

Equality

Because $\alpha > 1$ and $x, y > 0$, the coefficient

$$\lambda - 1 + \frac{(\alpha - 1)x^2}{\alpha x^2 + y^2}$$

is strictly positive. Hence equality in the k – th term occurs if and only if

$$x_k = x_{k+1}.$$

Equality in the sum therefore holds if and only if is true for every index k . With $x_{m+1} = x_1$, this is equivalent to

$$x_1 = x_2 = \dots = x_m > 0.$$

Conversely, if all the variables are equal, each first fraction is 1 and every squared-difference fraction is 0, so the left-hand side is exactly m . Thus the displayed family gives all equality cases.

JP.619 Let $a_1, a_2, \dots, a_n$ be nonzero real numbers such that

$a_1 + a_2 + a_n = 1$. Find the minimum value of

$$\sum_{k=1}^n \frac{a_k^3}{a_k^2 + a_k a_{k+1} + a_{k+1}^2}$$

where the subscripts are taken modulo n .

Proposed by José Luis Díaz – Barrero – Spain

Solution 1 by proposer

First observe that when $a_1 = a_2 = \dots = a_n = \frac{1}{n}$ the given sum equals $\frac{1}{3}$. So, we conjecture that

$$\sum_{k=1}^n \frac{a_k^3}{a_k^2 + a_k a_{k+1} + a_{k+1}^2} \geq \frac{1}{3}.$$

Indeed, on account of the constrain, instead of the preceding we will prove the stronger inequality

$$\sum_{k=1}^n \frac{a_k^3}{a_k^2 + a_k a_{k+1} + a_{k+1}^2} \geq \frac{a_1 + a_2 + \dots + a_n}{3}.$$

To do it, we consider the identity

$$\frac{a_1^3 - a_2^3}{a_1^2 + a_1 a_2 + a_2^2} + \frac{a_2^3 - a_3^3}{a_2^2 + a_2 a_3 + a_3^2} + \dots + \frac{a_n^3 - a_1^3}{a_n^2 + a_n a_1 + a_1^2} = (a_1 - a_2) + (a_2 - a_3) + \dots + (a_n - a_1) = 0,$$

from which it follows

$$\frac{a_1^3}{a_1^2 + a_1 a_2 + a_2^2} + \frac{a_2^3}{a_2^2 + a_2 a_3 + a_3^2} + \dots + \frac{a_n^3}{a_n^2 + a_n a_1 + a_1^2} = \frac{a_2^3}{a_1^2 + a_1 a_2 + a_2^2} + \frac{a_3^3}{a_2^2 + a_2 a_3 + a_3^2} + \dots + \frac{a_1^3}{a_n^2 + a_n a_1 + a_1^2}$$

Now, we have

$$\begin{aligned} & 2 \left(\frac{a_1^3}{a_1^2 + a_1 a_2 + a_2^2} + \frac{a_2^3}{a_2^2 + a_2 a_3 + a_3^2} + \dots + \frac{a_n^3}{a_n^2 + a_n a_1 + a_1^2} \right) \\ &= \frac{a_1^3 + a_2^3}{a_1^2 + a_1 a_2 + a_2^2} + \frac{a_2^3 + a_3^3}{a_2^2 + a_2 a_3 + a_3^2} + \dots + \frac{a_n^3 + a_1^3}{a_n^2 + a_n a_1 + a_1^2} \\ &= \frac{(a_1 + a_2)(a_1^2 - a_1 a_2 + a_2^2)}{a_1^2 + a_1 a_2 + a_2^2} + \dots + \frac{(a_n + a_1)(a_n^2 - a_n a_1 + a_1^2)}{a_n^2 + a_n a_1 + a_1^2} \\ &\geq \frac{1}{3} [(a_1 + a_2) + (a_2 + a_3) + \dots + (a_n + a_1)] = \frac{2}{3} (a_1 + a_2 + \dots + a_n) \end{aligned}$$

The last inequality is true on account that

$$\frac{a_k^2 - a_k a_{k+1} + a_{k+1}^2}{a_k^2 + a_k a_{k+1} + a_{k+1}^2} \geq \frac{1}{3}$$

as can be easily checked. Then, we have:

$$\sum_{k=1}^n \frac{a_k^3}{a_k^2 + a_k a_{k+1} + a_{k+1}^2} \geq \frac{1}{3},$$

with equality when $a_1 = a_2 = \dots = a_n = \frac{1}{n}$, and the conjecture is proven.

Finally, we may conclude that the minimum value of the given expression is $\frac{1}{3}$, and we are done.

Solution 2 by Marin Chirciu-Romania

Using $\frac{x^3}{x^2 + xy + y^2} \geq \frac{2x - y}{3} \Leftrightarrow (x - y)^2(x + y) \geq 0$ we obtain:

$$P = \sum_{k=1}^n \frac{a_k^3}{a_k^2 + a_k a_{k+1} + a_{k+1}^2} = \sum_{k=1}^n \frac{2a_k - a_{k+1}}{3} = \frac{1}{3} \sum_{k=1}^n a_k = \frac{1}{3} \cdot 1 = \frac{1}{3}$$

Equality holds if and only if $a_1 = a_2 = \dots = a_n = \frac{1}{n}$.

We deduce that $\min P = \frac{1}{3}$ for $a_1 = a_2 = \dots = a_n = \frac{1}{n}$.

Remark: The problem can be developed.

If $a_1, a_2, \dots, a_n \geq 0, \sum_{k=1}^n a_k = 1$ and $\lambda \geq 0$ then find min of

$$\sum_{k=1}^n \frac{a_k^3}{a_k^2 + \lambda a_k a_{k+1} + a_{k+1}^2}$$

Solution:

Using $\frac{x^3}{x^2 + \lambda xy + y^2} \geq \frac{2x-y}{\lambda+2} \Leftrightarrow (x-y)^2(\lambda x + y) \geq 0$ we obtain:

$$P = \sum_{k=1}^n \frac{a_k^3}{a_k^2 + \lambda a_k a_{k+1} + a_{k+1}^2} = \sum_{k=1}^n \frac{2a_k - a_{k+1}}{\lambda + 2} = \frac{1}{\lambda + 2} \sum_{k=1}^n a_k = \frac{1}{\lambda + 2} \cdot 1 = \frac{1}{\lambda + 2}.$$

Equality holds if and only if $a_1 = a_2 = \dots = a_n = \frac{1}{n}$.

We deduce that $\min P = \frac{1}{\lambda+2}$ for $a_1 = a_2 = \dots = a_n = \frac{1}{n}$.

Note: For $\lambda = 1$ we obtain Problem JP.619 from RMM Number 42 – Autumn 2026.

JP.619. If $a_1, a_2, \dots, a_n \geq 0, \sum_{k=1}^n a_k = 1$ then find min of

$$\sum_{k=1}^n \frac{a_k^3}{a_k^2 + a_k a_{k+1} + a_{k+1}^2}$$

Jose Luis Diaz – Barrero - Spain

JP.620 Determine the number of positive integers a such that $a|12!$ but 18 does not divide a .

Proposed by José Luis Díaz – Barrero – Spain

Solution by proposer

The prime factorization of $12!$ involves all prime numbers less than or equal to 12, which are $\{2, 3, 5, 7, 11\}$. Using Legendre's formula, the exponents are

$$\text{For } p = 2: \left\lfloor \frac{12}{2} \right\rfloor + \left\lfloor \frac{12}{4} \right\rfloor + \left\lfloor \frac{12}{8} \right\rfloor = 6 + 3 + 1 = 10 \Rightarrow 2^{10}$$

$$\text{For } p = 3: \left\lfloor \frac{12}{3} \right\rfloor + \left\lfloor \frac{12}{9} \right\rfloor = 4 + 1 = 5 \Rightarrow 3^5$$

$$\text{For } p = 5: \left\lfloor \frac{12}{5} \right\rfloor = 2 \Rightarrow 5^2$$

$$\text{For } p = 7: \left\lfloor \frac{12}{7} \right\rfloor = 1 \Rightarrow 7^1$$

$$\text{For } p = 11: \left\lfloor \frac{12}{11} \right\rfloor = 1 \Rightarrow 11^1,$$

$$\text{and } 12! = 2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^1 \cdot 11^1$$

Next, the prime factorization of 18 is $18 = 2^1 \cdot 3^2$. Any divisor a of $12!$ takes the general

$$\text{form } a = 2^{e_2} \cdot 3^{e_3} \cdot 5^{e_5} \cdot 7^{e_7} \cdot 11^{e_{11}}$$

where the exponents are bounded by:

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- $0 \leq e_2 \leq 10$
- $0 \leq e_3 \leq 5$
- $0 \leq e_5 \leq 2$
- $0 \leq e_7 \leq 1$
- $0 \leq e_{11} \leq 1$

The condition states that 18 does not divide $a(18 \nmid a)$. Since $18 = 2^1 \cdot 3^2$, for 18 to divide a , we would need to have at least one factor of 2 and at least two factors of 3. Therefore, 18 does divide a if and only if $e_2 \geq 1$ and $e_3 \geq 2$. Consequently, 18 does not divide a if and only if this condition fails, meaning

$$e_2 = 0 \quad \text{or} \quad e_3 < 2$$

To find the number of valid integers a , it is easier to find the total number of divisors of $12!$ and subtract the ones that are divisible by 18. We have

1. Total divisors of $12!$:

$$\text{Total} = (10 + 1)(5 + 1)(2 + 1)(1 + 1)(1 + 1) = 11 \times 6 \times 3 \times 2 \times 2 = 792$$

2. Divisors of $12!$ That are divisible by 18: For $18|a$, the exponents must satisfy $1 \leq e_2 \leq 10$ and $2 \leq e_3 \leq 5$ (while e_5, e_7, e_{11} have their normal ranges).

- Choices for e_2 : 10 choices (1 to 10)
- Choices for e_3 : 4 choices (2 to 5)
- Choices for e_5 : 3 choices (0 to 2)
- Choices for e_7 : 2 choices (0 to 1)
- Choices for e_{11} : 2 choices (0 to 1)

$$\text{Divisors divisible by 18} = 10 \times 4 \times 3 \times 2 \times 2 = 480$$

Finally, subtracting the invalid divisors from the total yields

$$\text{Valid divisors} = 792 - 480 = 312$$

JP.621 Determine all prime numbers p, q, r such that:

$$p^2 + q^2 - r(p + q) = 108$$

Proposed by José Luis Díaz – Barrero – Spain

Solution by proposer

Considering the equation modulo 2, we get

$$p^2 + q^2 - r(p - q) \equiv 108 \pmod{2}$$

Since $x^2 \equiv x \pmod{2}$ and $108 \equiv 0 \pmod{2}$, this simplifies to

$$(p + q) - r(p - q) \equiv 0 \pmod{2}$$

If one of the primes is 2 (let's say $p = 2$ and q is an odd prime), then $p - q$ and $p + q$ are both odd. The equation becomes:

$$\text{Odd} - r(\text{Odd}) \equiv 0 \pmod{2}$$

This requires r to be an odd prime.

Next, we analyze the cases where one of the variables is a small prime. We distinguish the following cases:

1. $p = 2$ and q is an odd prime. Substituting $p = 2$ into the original equation yields

$$2^2 + q^2 - r(2 - q) = 108 \text{ from which we get}$$

$$r = \frac{104 - q^2}{q - 2} = -(q + 2) + \frac{100}{q - 2}$$

For r to be an integer, $q - 2$ must be a positive divisor of 100.

The divisors of 100 are 1, 2, 4, 5, 10, 20, 25, 50, 100.

Testing $q - 2 = 5$, which gives $q = 7$ (a prime number)

$$r = -(7 + 2) + \frac{100}{5} = -9 + 20 = 11$$

Since 11 is a prime number, this yields a valid solution: $(p, q, r) = (2, 7, 11)$

2. Explicit check of other divisors. For completeness, we test the remaining positive divisors of 100 for $q - 2$, we have

- $q - 2 = 1 \Rightarrow q = 3, r = -5 + 100 = 95$ (Not prime)
- $q - 2 = 2 \Rightarrow q = 4$ (Not prime)
- $q - 2 = 4 \Rightarrow q = 6$ (Not prime)
- $q - 2 = 10 \Rightarrow q = 12$ (Not prime)
- $q - 2 = 20 \Rightarrow q = 22$ (Not prime)

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- $q - 2 = 25 \Rightarrow q = 27$ (Not prime)
- $q - 2 = 50 \Rightarrow q = 52$ (Not prime)
- $q - 2 = 100 \Rightarrow q = 102$ (Not prime)

3. Checking $q = 2$ with p as an Odd Prime. If we swap roles and set $q = 2$, the formula for r becomes

$$r = (p + 2) - \frac{100}{p - 2}$$

Testing the divisors of 100 for $p - 2$ yields

- $p - 2 = 1 \Rightarrow p = 3, r = 5 - 100 = -95$ (Negative, invalid)
- $p - 2 = 5 \Rightarrow p = 7, r = 9 - 20 = -11$ (Negative, invalid)
- All other divisors yields either non-prime values for p or negative values for r .

4. Checking $p = q$. If $p = q$, the term $r(p - q)$ vanishes:

$$p^2 + p^2 = 108 \Rightarrow 2p^2 = 108 \Rightarrow p^2 = 54$$

Since 54 is not a square, no solutions exist in this case.

Thus, the only set of prime numbers (p, q, r) satisfying the equation is

$$(p, q, r) = (2, 7, 11)$$

JP.622 Let a, b, c be positive reals such that $a + b + c = 3$. Find the minimum value of

$$\frac{a^2 - ab + b^2}{b + c} + \frac{b^2 - bc + c^2}{c + a} + \frac{c^2 - ca + a^2}{a + b}.$$

Proposed by José Luis Díaz – Barrero – Spain

Solution 1 by proposer

Since the given expression equals to $\frac{3}{2}$ when $a = b = c = 1$ then we conjecture that

$$\frac{a^2 - ab + b^2}{b + c} + \frac{b^2 - bc + c^2}{c + a} + \frac{c^2 - ca + a^2}{a + b} \geq \frac{3}{2}.$$

Indeed, for any real numbers x and y , we have the identity

$$x^2 - xy + y^2 = \frac{3}{4}(x - y)^2 + \frac{1}{4}(x + y)^2 \geq \frac{1}{4}(x + y)^2$$

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Applying this bound to each numerator in the given expression gives

$$\sum \frac{a^2 - ab + b^2}{b + c} \geq \frac{1}{4} \left(\frac{(a+b)^2}{b+c} + \frac{(b+c)^2}{c+a} + \frac{(c+a)^2}{a+b} \right)$$

By the Cauchy – Schwarz inequality, we have

$$\sum_{cyc} \frac{(a+b)^2}{b+c} \geq \frac{((a+b) + (b+c) + (c+a))^2}{(b+c) + (c+a) + (a+b)} = \frac{(2a+2b+2c)^2}{2(a+b+c)} = 2(a+b+c)$$

Substituting this back into our inequality yields

$$\sum_{cyc} \frac{a^2 - ab + b^2}{b+c} \geq \frac{1}{4} \cdot 2(a+b+c) = \frac{a+b+c}{2}$$

Since $a + b + c = 3$, then it follows that

$$\sum_{cyc} \frac{a^2 - ab + b^2}{b+c} \geq \frac{3}{2}$$

Solution 2 by Marin Chirciu-Romania

$$P = \sum \frac{a^2 - ab + b^2}{b+c} \stackrel{sos}{\geq} \sum \frac{a^2 + b^2}{2(b+c)} \stackrel{cs}{\geq} \frac{1}{2} \frac{(\sum a)^2 + (\sum b)^2}{\sum(b+c)} = \frac{1}{2} \frac{2(\sum a)^2}{2\sum a} = \frac{1}{2} \sum a = \frac{3}{2}$$

Equality holds if and only if $a = b = c = 1$.

We deduce that $\min P = \frac{3}{2}$ for $(a, b, c) = (1, 1, 1)$.

Remark: The problem can be developed.

If $a, b, c > 0, \sum a = 3$ and $\lambda \leq 2$ then find min of

$$\sum \frac{a^2 - \lambda ab + b^2}{b+c}$$

Marin Chirciu

Solution:

$$\begin{aligned} P &= \sum \frac{a^2 - \lambda ab + b^2}{b+c} \stackrel{sos}{\geq} \sum \frac{(2-\lambda)(a^2 + b^2)}{2(b+c)} \stackrel{cs}{\geq} \frac{2-\lambda}{2} \cdot \frac{(\sum a)^2 + (\sum b)^2}{\sum(b+c)} = \\ &= \frac{2-\lambda}{2} \cdot \frac{2(\sum a)^2}{2\sum a} = \frac{2-\lambda}{2} \cdot \sum a = \frac{3}{2}(2-\lambda). \end{aligned}$$

Equality holds if and only if $a = b = c = 1$.

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We deduce that $\min P = \frac{3}{2}(2 - \lambda)$ for $(a, b, c) = (1, 1, 1)$.

Note: For $\lambda = 1$ we obtain Problem JP.622 from RMM Number 42 – Autumn 2026.

JP.622. If $a, b, c > 0, \sum a = 3$ then find min of

$$\sum \frac{a^2 - ab + b^2}{b + c}$$

Jose Luis Diaz – Barrero – Spain

JP.623 Find all values of n in the range $1 \leq n \leq 10$ for which the sum

$$S = \sum_{k=1}^{n^2+n} [\sqrt{k}]$$

is a prime number. Here $[x]$ represents the integer part of x .

Proposed by José Luis Díaz – Barrero – Spain

Solution by proposer

Let $M = n^2 + n$. We evaluate the sum by grouping terms according to the value of $j = [\sqrt{k}]$. The condition $[\sqrt{k}] = j$ is equivalent to

$$j^2 \leq k < (j+1)^2$$

Since $M = n^2 + n$ satisfies $n^2 \leq n^2 + n < (n+1)^2$, the maximum integer value for j is n . For each j from 1 to $n-1$, the number of terms is $(j+1)^2 - j^2 = 2j+1$. For $j = n$, the terms run from n^2 to $n^2 + n$, giving a total of $(n^2 + n) - n^2 + 1 = n+1$ terms. Thus, the sum can be expressed as

$$\begin{aligned} S &= \sum_{j=1}^{n-1} j(2j+1) + n(n+1) = \sum_{j=1}^{n-1} (2j^2 + j) + n(n+1) \\ &= 2 \left(\frac{(n-1)n(2n-1)}{6} \right) + \frac{(n-1)n}{2} + n(n+1) \end{aligned}$$

from which we get the closed form polynomial.

$$S = \frac{n(4n^2 + 3n + 5)}{6}$$

Substituting integers n from 1 to 10 into the expression yields

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For $n = 1$: $\frac{1(4(1)^2+3(1)+5)}{6} = 2$ (Prime)

For $n = 2$: $\frac{2(4(2)^2+3(2)+5)}{6} = 9$ (Composite)

For $n = 3$: $\frac{3(4(3)^2+3(3)+5)}{6} = 25$ (Composite)

For $n = 4$: $\frac{4(4(4)^2+3(4)+5)}{6} = 54$ (Composite)

For $n = 5$: $\frac{5(4(5)^2+3(5)+5)}{6} = 100$ (Composite)

For $n = 6$: $\frac{6(4(6)^2+3(6)+5)}{6} = 167$ (Prime)

For $n = 7$: $\frac{7(4(7)^2+3(7)+5)}{6} = 259$ (Composite)

For $n = 8$: $\frac{8(4(8)^2+3(8)+5)}{6} = 380$ (Composite)

For $n = 9$: $\frac{9(4(9)^2+3(9)+5)}{6} = 534$ (Composite)

For $n = 10$: $\frac{10(4(10)^2+3(10)+5)}{6} = 725$ (Composite)

Finally, we conclude that the values of n in the range $1 \leq n \leq 10$ for which the sum is a prime number are $n = 1$ and $n = 6$.

JP.624 If $a, b, c > 0, a + b + c = 3$ and $0 \leq \lambda \leq 2$ then:

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} - \frac{\lambda}{abc} \leq 1$$

Proposed by Marin Chirciu – Romania

Solution by proposer

$$\begin{aligned} \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} - \frac{\lambda}{abc} &= \frac{1}{3} \sum \frac{1}{a^2} + \frac{2}{3} \sum \frac{1}{a^2} - \frac{\lambda}{abc} \cdot \frac{a+b+c}{3} = \\ &= \frac{1}{3} \left[\sum \left(\frac{1}{a^2} + a + a \right) - 2 \sum a \right] + \frac{2-\lambda}{3} \sum \frac{1}{a^2} + \frac{\lambda}{3} \left(\sum \frac{1}{a^2} - \sum \frac{1}{bc} \right) \stackrel{AGM}{\geq} \\ &\stackrel{AGM}{\geq} \frac{1}{3} \left[\sum 3 \sqrt[3]{\frac{1}{a^2} \cdot a \cdot a} - 2 \cdot 3 \right] + \frac{2-\lambda}{3} \sum \frac{1}{a^2} + \frac{\lambda}{3} \left(\sum \frac{1}{a^2} - \sum \frac{1}{bc} \right) = \\ &= \frac{1}{3} \left(\sum 3 \sqrt[3]{1} - 2 \cdot 3 \right) + \frac{2-\lambda}{3} \sum \frac{1}{a^2} + \frac{\lambda}{6} \sum \left(\frac{1}{b} - \frac{1}{c} \right)^2 \stackrel{0 \leq \lambda \leq 2}{=} \end{aligned}$$

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$$= \frac{1}{3}(9 - 6) + 0 \cdot \sum \frac{1}{a^2} + 0 = 1.$$

Equality holds if and only if $a = b = c = 1$ and $\lambda = 2$.

JP.625 If $a, b, c \geq 0, a + b + c = 2$ and $\lambda \geq 1$ then:

$$\sum \sqrt{\frac{b+c}{a^2+\lambda}} \geq \frac{4}{\sqrt{\lambda+1}}$$

Proposed by Marin Chirciu – Romania

Solution by proposer

We prove: Lemma: If $0 \leq x \leq 2$ and $\lambda \geq 1$ then:

$$\sqrt{\frac{2-x}{x^2+\lambda}} \geq \frac{2-x}{\sqrt{1-\lambda}}$$

Proof: We have $\sqrt{\frac{2-x}{x^2+\lambda}} \geq \frac{2-x}{\sqrt{1-\lambda}} \Leftrightarrow x(2-x)(x-1)(x^2-x+\lambda-1) \geq 0$, which follows

from $\lambda \geq 1$ with equality for $x \in \{0, 1, 2\}$. Let's get back to the main problem:

Using the Lemma we obtain:

$$\begin{aligned} LHS &= \sum \sqrt{\frac{b+c}{a^2+\lambda}} = \sum \sqrt{\frac{2-a}{a^2+\lambda}} \stackrel{\text{Lemma}}{\geq} \sum \frac{2-a}{\sqrt{\lambda+1}} = 3 \cdot \frac{2}{\sqrt{\lambda+1}} - \frac{1}{\sqrt{\lambda+1}} \sum a = \\ &= 3 \cdot \frac{2}{\sqrt{\lambda+1}} - \frac{1}{\sqrt{\lambda+1}} \cdot 2 = \frac{4}{\sqrt{\lambda+1}} = RHS \end{aligned}$$

JP.626 If $a, b, c > 0, a^n + b^n + c^n = 3$ and $n \in \mathbb{N}^*$ then:

$$\sum \frac{a}{a^{2n+1} + b^{2n} + c^{2n-1}} \leq 1$$

Proposed by Marin Chirciu – Romania

Solution by proposer

We prove: Lemma: If $a, b, c > 0, a^n + b^n + c^n = 3$ then:

$$\frac{a}{a^{2n+1} + b^{2n} + c^{2n-1}} \leq \frac{1+a+ac}{9}$$

Using CBS inequality we obtain:

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$$(a^{2n+1} + b^{2n} + c^{2n-1}) \left(\frac{1}{a} + 1 + c \right) \geq (a^n + b^n + c^n)^2 = 9 \Leftrightarrow$$

$$\Leftrightarrow \frac{a}{a^{2n+1} + b^{2n} + c^{2n-1}} \leq \frac{1 + a + ac}{9}.$$

Let's get back to the main problem. Using the Lemma, we obtain:

$$LHS = \sum \frac{a}{a^{2n+1} + b^{2n} + c^{2n-1}} \stackrel{\text{Lemma}}{\leq} \sum \frac{1 + a + ac}{9} = \frac{1}{9} \left(3 + \sum a + \sum ac \right) \stackrel{(1)}{\leq}$$

$$\stackrel{(1)}{\leq} \frac{1}{9} (3 + 3 + 3) = 1 = RHS,$$

where (1) $\Leftrightarrow \sum ac \leq 3$, which follows from $\sum bc \leq \frac{1}{3} (\sum a)^2 \stackrel{(2)}{\leq} \frac{1}{3} (3)^2 = 3$,

where (2) $\Leftrightarrow \sum a \leq 3$, which follows from Holder's inequality:

$$\sum a^n \stackrel{\text{Holder}}{\geq} \frac{(\sum a)^n}{3^{n-1}}, \sum a^n = 3 \Rightarrow 3 \stackrel{\text{Holder}}{\geq} \frac{(\sum a)^n}{3^{n-1}} \Rightarrow \sum a \leq 3$$

Equality holds if and only if $a = b = c = 1$.

JP.627 If $a, b, c > 0$ and $\lambda \geq 0$ then:

$$\sum \frac{\sqrt{a^2 + (b + \lambda c)^2}}{b + \lambda c} \geq \frac{6\lambda + 9}{(\lambda + 1)\sqrt{5}}$$

Proposed by Marin Chirciu – Romania

Solution by proposer

We prove: Lemma: If $a, b, c > 0$ and $\lambda \geq 0$ then:

$$\frac{\sqrt{a^2 + (b + 2\lambda c)^2}}{b + \lambda c} \geq \frac{1}{\sqrt{5}} \left(2 + \frac{a}{b + \lambda c} \right)$$

Solution:

$$\frac{\sqrt{a^2 + (b + \lambda c)^2}}{b + \lambda c} = \sqrt{\frac{a^2}{(b + \lambda c)^2} + 1} = \sqrt{t^2 + 1} \stackrel{(1)}{\geq} \frac{1}{\sqrt{5}} (t + 2), \text{ where (1) } \Leftrightarrow \sqrt{t^2 + 1} \geq \frac{1}{\sqrt{5}} (t + 2) \Leftrightarrow$$

$$\Leftrightarrow 5(t^2 + 1) \geq (t + 2)^2 \Leftrightarrow (2t - 1)^2 \geq 0, \text{ obviously with equality for } t = \frac{1}{2} \Leftrightarrow$$

$$\Leftrightarrow \frac{a}{b + \lambda c} = \frac{1}{2} \Leftrightarrow b + \lambda c = 2a$$

Let's get back to the main problem. Using the Lemma, we obtain:

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$$\begin{aligned} LHS &= \sum \frac{\sqrt{a^2 + (b + \lambda c)^2}}{b + \lambda c} \stackrel{\text{Lemma}}{\geq} \sum \frac{1}{\sqrt{5}} \left(2 + \frac{a}{b + \lambda c} \right) = \frac{1}{\sqrt{5}} \left(6 + \sum \frac{a}{b + \lambda c} \right) \stackrel{\text{Nesbitt}}{\geq} \\ &\geq \frac{1}{\sqrt{5}} \left(6 + \frac{3}{\lambda + 1} \right) = \frac{1}{\sqrt{5}} \left(\frac{6\lambda + 9}{\lambda + 1} \right) = \frac{6\lambda + 9}{(\lambda + 1)\sqrt{5}} = RHS \end{aligned}$$

Equality holds if and only if $a = b = c = \lambda = 1$.

JP.628 How many ways are there to put 15 balls into four bins so that the first two bins have between 2 and 6 balls, the third bin has at least 5 balls and the last bin has an odd number of balls?

Proposed by José Luis Díaz – Barrero – Spain

Solution by proposer

To solve the problem of finding the number of ways to distribute 15 identical balls into four bins subject to the given constraints, we model it as finding the number of integer solutions (x_1, x_2, x_3, x_4) to the equation

$$x_1 + x_2 + x_3 + x_4 = 15$$

subject to the constraints

1. $2 \leq x_1 \leq 6$
2. $2 \leq x_2 \leq 6$
3. $x_3 \geq 5$
4. $x_4 \geq 1$ and x_4 is odd

To convert these inequalities into non-negative integer variables, let

$$y_1 = x_1 - 2 \Rightarrow 0 \leq y_1 \leq 4$$

$$y_2 = x_2 - 2 \Rightarrow 0 \leq y_2 \leq 4$$

$$y_3 = x_3 - 5 \Rightarrow y_3 \geq 0$$

$$x_4 = 2y_4 + 1 \Rightarrow y_4 \geq 0 \text{ (since } x_4 \text{ is an odd positive integer)}$$

Substituting these into the original equation yields

$$(y_1 + 2) + (y_2 + 2) + (y_3 + 5) + (2y_4 + 1) = 15$$

or $y_1 + y_2 + y_3 + 2y_4 = 5$. Since y_4 is a non-negative integer, $2y_4$ can take the values 0, 2 or 4 (corresponding to $y_4 = 0, 1, 2$). We consider the following cases:

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1. $y_4 = 0$ ($2y_4 = 0$). The equation becomes

$$y_1 + y_2 + y_3 = 5$$

with $0 \leq y_1 \leq 4$ and $0 \leq y_2 \leq 4$.

- Without upper bounds, the total number of non-negative integer solutions is

$$\binom{5+3-1}{3-1} = \binom{7}{2} = 21$$

- We subtract invalid solutions where $y_1 \geq 5$ or $y_2 \geq 5$

- If $y_1 \geq 5$, the only solution is $(y_1 = 5, y_2 = 0, y_3 = 0)$ (1 solution)

- If $y_2 \geq 5$, the only solution is $(y_1 = 0, y_2 = 5, y_3 = 0)$ (1 solution)

Number of valid solutions for Case 1:

$$21 - 1 - 1 = 19$$

2. $y_4 = 1$ ($2y_4 = 2$). The equation becomes

$$y_1 + y_2 + y_3 = 3$$

with $0 \leq y_1 \leq 4$ and $0 \leq y_2 \leq 4$. All solutions naturally satisfy the upper bounds since the maximum value is 3.

$$\binom{3+3-1}{3-1} = \binom{5}{2} = 10$$

3. $y_4 = 2$ ($2y_4 = 4$). The equation becomes

$$y_1 + y_2 + y_3 = 1$$

with $0 \leq y_1 \leq 4$ and $0 \leq y_2 \leq 4$. All solutions naturally satisfy the upper bounds.

$$\binom{1+3-1}{3-1} = \binom{3}{2} = 3$$

Finally, summing the valid solutions from all three cases, we obtain

$$\text{Total ways} = 19 + 10 + 3 = 32$$

JP.629 The numbers 1, 2, ..., 1000 are written on the board. Two numbers are erased and replaced by the remainder of their sum when divided by 19. This operation is repeated until one number is left. What is the number?

Proposed by José Luis Díaz – Barrero – Spain

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Solution by proposer

To find the final remaining number when the numbers $1, 2, \dots, 1000$ are repeatedly replaced by the remainder of their sum when divided by 19, we can use the concept of an invariant.

When two numbers, a and b , are erased and replaced by $(a + b)(\text{mod } 19)$, the change in the total sum of all numbers on the board is

$$\text{New Sum} = \text{Old Sum} - a - b + (a + b) (\text{mod } 19)$$

Because $(a + b) (\text{mod } 19)$ is congruent to $(a + b)$ modulo 19, the sum of all numbers on the board modulo 19 remains invariant at every single step. No matter which pairs are chosen or in what order, the final remaining single number will be congruent modulo 19 to the sum of the initial numbers.

The initial numbers form an arithmetic progression from 1 to 1000 which sum is

$$\text{Sum} = \frac{n(n+1)}{2} = \frac{1000 \times 1001}{2} = 500 \times 1001 = 500,500$$

Now, we compute $500,500 (\text{mod } 19)$ and we get $500,500 = 19 \times 26,342 + 2$

Thus, the remainder when the total sum is divided by 19 is 2 and the final remaining number on the board is 2.

JP.630 In Imaginaryland there are 2027 towns. Every pair of towns is either connected by a single road, or is not connected. If we consider any subset of 2025 towns, the total number of roads connecting these towns to each other is a constant. If there are R roads in Imaginaryland, then find all possible values of R .

Proposed by José Luis Díaz – Barrero – Spain

Solution by proposer

We consider the general case with n towns. Let K denote the constant number of roads connecting any subset of $n - 2$ towns and let $c_{ij} \in \{0, 1\}$ denote the number roads

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connecting town i and town j . Finally, for $i = 1, 2, \dots, n$ let d_i denote the total number of roads connected to town i . Note that

$$R \leq \binom{n}{2} = \frac{n(n-1)}{2}$$

Clearly,

$$\sum_{i=1}^n d_i = 2R \quad \text{and} \quad \sum c_{ij} = R,$$

Where the latter sum is over all 2 – element subsets $\{i, j\}$ of $\{1, 2, \dots, n\}$. The number of roads connected to at least one of the towns with number i or j is equal to $d_i + d_j - c_{ij}$.

Thus, for any 2 – element subsets $\{i, j\} \subset \{1, 2, \dots, n\}$, we have

$$K = R - d_i - d_j + c_{ij}$$

Adding all these equations for every 2 – element subset $\{i, j\}$ yields

$$\binom{n}{2} K = \binom{n}{2} R - 2(n-1)R + R$$

which may be written as

$$n(n-1)K = (n-2)(n-3)R$$

Note that both $n(n-1)$ and $(n-2)(n-3)$ are divisible by 2 and that the only integer $k > 2$ which divides both $n(n-1)$ and $(n-2)(n-3)$ is 3, this latter case occurring if and only if n is divisible by 3. Since 3 does not divide 2027, in the situation of the given problem $\frac{n(n-1)}{2}$ and $\frac{(n-2)(n-3)}{2}$ are coprime. Hence, R is a multiple of $\frac{n(n-1)}{2}$. As $R \leq \frac{n(n-1)}{2}$

with equality when all the pairs of towns are connected the only possibility is $R = \frac{n(n-1)}{2}$.

Therefore, the total number of roads is

$$R = 2027 \times 1013 = 2053351$$

PROBLEMS FOR SENIORS

SP.616 Let a, b, c, d be nonnegative real numbers such that at most one of the is less than 1 and $ab + ac + ad + bc + bd + cd = 6$. Prove that:

$$(2a^2 + 1)(2b^2 + 1)(2c^2 + 1)(2d^2 + 1) \geq 81.$$

Proposed by Vasile Cîrtoaje – Romania

Solution by proposer

The equality constraint implies the known inequality

$$a + b + c + d \geq 4.$$

Consider

$$a = \min\{a, b, c, d\} \leq 1, \quad b, c \geq 1, \quad d = \max\{a, b, c, d\} \geq 1,$$

and write the required inequality as $E \geq 0$, where

$$E = \ln(2a^2 + 1) + \ln(2b^2 + 1) + \ln(2c^2 + 1) + \ln(2d^2 + 1) - 4 \ln 3.$$

For fixed c and d , assume that a and E are functions of b . We have

$$(b + c + d)a' + a + c + d = 0 \text{ and}$$

$$\frac{E'(b)}{4} = \frac{b}{2b^2 + 1} + \frac{aa'}{2a^2 + 1} = \frac{b}{2b^2 + 1} - \frac{a(a + c + d)}{(2a^2 + 1)(b + c + d)}.$$

We claim that $E'(b) \geq 0$. Write this inequality as follows:

$$\begin{aligned} \frac{b + c + d}{a + c + d} &\geq \frac{a(2b^2 + 1)}{b(2a^2 + 1)}, \quad \frac{b + c + d}{a + c + d} - 1 \geq \frac{a(2b^2 + 1)}{b(2a^2 + 1)} - 1, \\ \frac{b - a}{a + c + d} &\geq \frac{(b - a)(2ab - 1)}{b(2a^2 + 1)}, \quad (b - a)[a + b + c + d - 2ab(c + d)] \geq 0. \end{aligned}$$

We need to show that $a + b + c + d \geq 2ab(c + d)$. Since $a + b + c + d \geq 4$ and

$ab \leq a + b - 1$, it suffices to show that $2 \geq (a + b - 1)(c + d)$, that is

$$2 + c + d \geq (a + b)(c + d), \quad 2 + c + d \geq 6 - a - cd, \quad ab + cd \geq 4 - c - d.$$

Since $4 - c \leq a + b$, it suffices to show that $ab + cd \geq a + b$. Indeed,

$$ab + cd - a - b = a(b - 1) + cd - b \geq cd - b \geq d - b \geq 0.$$

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From $E'(b) \geq 0$, it follows that $E(b)$ is increasing and has the minimum value when b is minimum, hence when $b = 1$. So, it suffices to prove the inequality $E \geq 0$ for $b = 1$.

Similarly, it suffices to prove the inequality $E \geq 0$ for $c = 1$, hence for $b = c = 1$. We need to show that $ad + 2(a + d) = 5$ implies $F \geq 0$, where $F = (2a^2 + 1)(2d^2 + 1) - 9$. We have

$$F = 4a^2d^2 + 2(a + d)^2 - 4ad - 8,$$

hence

$$2F = 8a^2d^2 + (5 - ad)^2 - 8ad - 16 = 9(ad - 1)^2 \geq 0.$$

So, the proof is finished. Equality occurs for $(a, b, c, d) = (1, 1, 1, 1)$.

SP.617 Let a, b, c, d be positive real numbers such that $a \geq b \geq 1 \geq c \geq d$ and $ab + ac + ad + bc + bd + cd = 6$. Prove that:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + 8 \geq 3(a + b + c + d)$$

Proposed by Vasile Cîrtoaje – Romania

Solution by proposer

Write the inequality as $E \geq 0$, where

$$E = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + 8 - 3(a + b + c + d).$$

For fixed a and b , we may consider that c and E are functions of d . By differentiating the equality constraint, we get $(a + b + d)c' + a + b + c = 0$. So,

$$E'(d) = \left(-\frac{1}{c^2} - 3\right)c' - \frac{1}{d^2} - 3 = \left(\frac{1}{c^2} + 3\right)\frac{a + b + c}{a + b + d} - \frac{1}{d^2} - 3.$$

We will show that $E'(d) \leq 0$, that is

$$\frac{a + b + c}{a + b + d} \leq \frac{\frac{1}{d^2} + 3}{\frac{1}{c^2} + 3}, \frac{a + b + c}{a + b + d} - 1 \leq \frac{\frac{1}{d^2} + 3}{\frac{1}{c^2} + 3} - 1, \frac{c - d}{a + b + d} \leq \frac{(c - d)(c + d)}{(3c^2 + 1)d^2}.$$

We need to show that

$$\frac{1}{a + b + d} \leq \frac{c + d}{(3c^2 + 1)d^2}.$$

It is true because

$$\frac{1}{a+b+d} \leq \frac{1}{1+1+d} \leq \frac{1}{2}$$

and

$$\frac{c+d}{(3c^2+1)d^2} \geq \frac{2d}{4d^2} = \frac{1}{2d} \geq \frac{1}{2}.$$

Since $E'(d) \leq 0$, $E(d)$ is decreasing and has the minimum value when d is maximum, hence when $d = c$. Therefore, it suffices to consider this case, when we need to show that $F \geq 0$ for $a \geq b \geq 1 \geq c$ and $ab + 2ac + 2bc + c^2 = 6$, where

$$F = \frac{1}{a} + \frac{1}{b} + \frac{2}{c} + 8 - 3(a+b+2c).$$

For fixed a , assume that c and F are functions of b . We have $2(a+b+c)c' + a + 2c = 0$ and

$$F'(b) = \left(-\frac{2}{c^2} - 6\right) - \frac{1}{b^2} - 3 = \left(\frac{1}{c^2} + 3\right) \frac{a+2c}{a+b+c} - \frac{1}{b^2} - 3.$$

We will show that $F'(b) \geq 0$, that is

$$\frac{\frac{1}{c^2} + 3}{\frac{1}{b^2} + 3} \geq \frac{a+b+c}{a+2c}, \frac{\frac{1}{c^2} + 3}{\frac{1}{b^2} + 3} - 1 \geq \frac{a+b+c}{a+2c} - 1, \frac{(b-c)(b+c)}{(3b^2+1)c^2} \geq \frac{b-c}{a+2c}.$$

We need to show that $(b+c)(a+2c) \geq (3b^2+1)c^2$. Since

$$\begin{aligned} (b+c)(a+2c) - (3b^2+1)c^2 &> (b+c)^2 - (3b^2+1)c^2 = b(b+2c-3bc^2) = \\ &= b^2(1-c^2) + 2bc(1-bc), \end{aligned}$$

it suffices to show that $bc \leq 1$. Indeed, from

$$6 = a(b+2c) + 2bc + c^2 \geq b(b+2c) + 2bc + c^2 = b^2 + c^2 + 4bc \geq 6bc,$$

we obtain $bc < 1$. Since $F'(b) > 0$, $F(b)$ is increasing and has the minimum value when b is minimum, hence when $b = 1$. Therefore, it is enough to consider $b = 1$, when we need to show that

$$\frac{1}{a} + \frac{2}{c} - 3 \geq 3(a+2c-3)$$

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for $a \geq 1 \geq c$ and $a + 2ac + 2c + c^2 = 6$. Since $a = \frac{6-2c-c^2}{2c+1}$, the inequality is equivalent to

$$\frac{(c-1)^2(c+4)}{c(6-2c-c^2)} \geq \frac{3(c-1)^2}{2c+1}, \quad (c-1)^2(c^3+4c^2+3c+4) \geq 0.$$

The proof is finished. The equality occurs for $a = b = c = d = 1$.

SP.618 Prove that $\frac{17}{3}$ is the smallest positive value of the constant k such that

$$\frac{1}{a+k} + \frac{1}{b+k} + \frac{1}{c+k} + \frac{1}{d+k} \leq \frac{4}{1+k}$$

for any $a, b, c, d \geq \frac{1}{16}$ with $ab + ac + ad + bc + bd + cd = 6$.

Proposed by Vasile Cîrtoaje – Romania

Solution by proposer

For $a = b = c = \frac{4}{3}$ and $d = \frac{1}{6}$, the constraints are satisfied and the inequality implies $k \geq \frac{17}{3}$. To show that $\frac{17}{3}$ is the smallest positive value of k , we need to prove that $E \leq 0$, where

$$E = \frac{1}{3a+17} + \frac{1}{3b+17} + \frac{1}{3c+17} + \frac{1}{3d+17} - \frac{1}{5}.$$

For fixed c and d , the expression E has the maximum value when

$$F(a, b) = \frac{1}{3a+17} + \frac{1}{3b+17}$$

is maximum. Assuming $a \geq b$ and denoting

$$a_1 = c + d, \quad b_1 = 6 - cd,$$

we have $a_1 > 0, b_1 > 0$ and $ab + a_1(a+b) = b_1$. By Lemma below, the expression

$F(a, b)$ has the maximum value when $b = \frac{1}{6}$ or $a = b$. So, it suffices to prove that $E \leq 0$

for these two cases. Similarly, applying Lemma for any two of the numbers a, b, c, d with

$a, b, c, d \geq \frac{1}{6}$, it suffices to prove that $E \leq 0$ for

$$(a, b, c, d) = (1, 1, 1, 1), (a, b, c, d) = \left(x, x, x, \frac{1}{6}\right),$$

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$(a, b, c, d) = (y, y, \frac{1}{6}, \frac{1}{6})$, and $(a, b, c, d) = (z, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$, where $x = \frac{4}{3}$, $y = \frac{\sqrt{219}-2}{6} \approx 2.1331$ (with $36y^2 + 24y - 215 = 0$), and $z = \frac{71}{6}$. In the first two cases, we have $E = 0$. In the third case, we have

$$E(y) = \frac{2}{3y+17} - \frac{3}{35}.$$

Since $y > \frac{17}{8}$, we get

$$E(y) < E\left(\frac{17}{8}\right) = \frac{16}{187} - \frac{3}{35} = \frac{-1}{6545} < 0.$$

In the last case, we have

$$E(z) = \frac{1}{3z+17} - \frac{1}{35} = \frac{2}{105} - \frac{1}{35} = \frac{-1}{105} < 0.$$

The proof is finished. If $a \geq b \geq c \geq d \geq \frac{1}{6}$, then the equality occurs for

$$(a, b, c, d) = (1, 1, 1, 1) \text{ and } (a, b, c, d) = \left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}, \frac{1}{6}\right).$$

Lemma. Let a_1, b_1 and m be positive real constants, and let $a \geq b \geq m$ such that

$$ab + a_1(a+b) = b_1.$$

For $k > 0$, the expression

$$E = \frac{1}{a+k} + \frac{1}{b+k}$$

has extreme values for $a = b$ or $b = m$.

Proof. Denoting $s = \frac{a+b}{2}$, the equality constraint becomes $ab + 2a_1s = b_1$. From

$$b_1 = ab + 2a_1s \leq s^2 + 2a_1s,$$

it follows that s is minimum when $ab = s^2$, hence when $a = b$. From $0 \leq (a-m)(b-m)$ (with equality for $b = m$), we get $0 \leq ab - 2ms + m^2 - b_1 - 2a_1s - 2ms + m^2$, hence

$s \leq \frac{b_1+m^2}{2(a_1+m)}$; so, s is maximum when $b = m$. As a consequence, it suffices to show that the expression

$$E = \frac{2(s+k)}{b_1 + 2(k-a_1)s + k^2}$$

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has extreme values when s is minimum or maximum. From

$$E'(s) = \frac{2(2ka_1 + b_1 - k^2)}{[b_1 + 2(k - a_1)s + k^2]^2},$$

it follows that $E(s)$ is increasing for $2ka_1 + b_1 - k^2 \geq 0$, and decreasing for

$2ka_1 + b_1 - k^2 \leq 0$. In both cases, $E(s)$ has extreme values when s is extreme, hence when $a = b$ or $b = m$.

SP.619 Let a, b, c, d, e be positive real numbers such that $a \geq b \geq c \geq d \geq e$ and $abcde = 1$. Prove that:

$$\frac{1}{3ab+2} + \frac{1}{3bc+2} + \frac{1}{3cd+2} + \frac{1}{3de+2} + \frac{1}{3ea+2} \geq 1$$

Proposed by Vasile Cîrtoaje – Romania

Solution by proposer

Write the inequality as $E \geq 0$, where

$$E = \frac{1}{3ab+2} + \frac{1}{3bc+2} + \frac{1}{3cd+2} + \frac{1}{3de+2} + \frac{1}{3ea+2} - 1$$

For fixed a, d and e , we may consider that b and E are functions of c . Since

$$bc = \frac{1}{ade} = \text{fixed}, \text{ we have } cb' + b = 0 \text{ and}$$

$$\begin{aligned} \frac{E'(c)}{3} &= \frac{-ab'}{(3ab+2)^2} - \frac{d}{(3cd+2)^2} = \frac{ab}{c(3ab+2)^2} - \frac{d}{(3cd+2)^2} = \\ &= \frac{(ab-cd)(4-9abcd)}{c(3ab+2)^2(3cd+2)^2} \leq 0 \end{aligned}$$

So, $E(c)$ is decreasing and has the minimum value when c is maximum, hence when $c = b$. Therefore, it suffices to prove the required inequality for $c = b$, when we need to

show that

$$\frac{1}{3ab+2} + \frac{1}{3b^2+2} + \frac{1}{3bd+2} + \frac{1}{3de+2} + \frac{1}{3ae+2} \geq 1$$

for $a \geq b \geq d \geq e$ such that $ab^2de = 1$. Denote $x = de$ and $y = bd$. Since $ab = \frac{1}{bx}$ and

$$ae = \frac{1}{by} \text{ the inequality becomes}$$

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$$\frac{bx}{2bx+3} + \frac{1}{3b^2+2} + \frac{1}{3y+2} + \frac{1}{3x+2} + \frac{by}{2by+3} \geq 1,$$

that is

$$f(x) + f(y) + \frac{1}{3b^2+2} \geq 1,$$

$$\text{where } f(t) = \frac{bt}{2bt+3} + \frac{1}{3t+2}$$

Case 1: $b \geq \frac{9}{4}$. By Lemma below, we have $f(x) \geq \frac{1}{2}$ and $f(y) \geq \frac{1}{2}$. So, the required inequality is true.

Case 2: $1 \leq b \leq \frac{9}{4}$. By Lemma, we have

$$f(x) \geq \frac{2\sqrt{b}}{2\sqrt{b}+3}, \quad f(y) \geq \frac{2\sqrt{b}}{2\sqrt{b}+3}.$$

So, it suffices to show that

$$\frac{4\sqrt{b}}{2\sqrt{b}+3} + \frac{1}{3b^2+2} \geq 1.$$

Denoting $z = \sqrt{b} \geq 1$, the inequality becomes

$$\frac{4z}{2z+3} \geq \frac{3z^4+1}{3z^4+2}, \quad 2z^5 - 3z^4 + 2z - 1 \geq 0, \quad (z-1)^2(2z^3+z^2-1) \geq 0.$$

Case 3: $0 < b \leq 1$. Since $x = de \leq b^2$ and $y = bd \leq b^2$, by Lemma below we have

$$f(x) \geq \frac{b^3}{2b^3+3} + \frac{1}{3b^2+2}, \quad f(y) = \frac{b^3}{2b^3+3} + \frac{1}{3b^2+2}.$$

So, it suffices to show that

$$\frac{2b^3}{2b^3+3} + \frac{3}{3b^2+2} \geq 1$$

which is equivalent to

$$\frac{2b^3}{2b^3+3} \geq \frac{3b^2-1}{3b^2+2}, \quad 2b^3 - 3b^2 + 1 \geq 0, \quad (b-1)^2(2b+1) \geq 0.$$

Thus, the proof is completed. The equality occurs for $a = b = c = d = e = 1$.

Lemma. Let

$$f(t) = \frac{bt}{2bt+3} + \frac{1}{3t+2}, \quad t \geq 0.$$

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(a) For $b \geq \frac{9}{4}$, we have $f(t) \geq \frac{1}{2}$.

(b) For $1 \leq b \leq \frac{9}{4}$, we have $f(t) \geq f\left(\frac{1}{\sqrt{b}}\right) = \frac{2\sqrt{b}}{2\sqrt{b}+3}$.

(c) For $0 < b \leq 1$ and $0 \leq t \leq b^2$, we have $f(t) \geq f(b^2) = \frac{b^3}{2b^3+3} + \frac{1}{3b^2+2}$.

Proof.

We have

$$\begin{aligned} \frac{f'(t)}{3} &= \frac{b}{(2bt+3)^2} - \frac{1}{(3t+2)^2} = \frac{[\sqrt{b}(3t+2) - 2bt - 3][\sqrt{b}(3t+2) + 2bt + 3]}{(2bt+3)^2(3t+2)^2} \\ &= \frac{(3 - 2\sqrt{b})(\sqrt{b} \cdot t - 1)[\sqrt{b}(3t+2) + 2bt + 3]}{(2bt+3)^2(3t+2)^2}. \end{aligned}$$

(a) We have

$$f(t) - \frac{1}{2} = \frac{bt}{2bt+3} - \frac{3t}{2(3t+2)} = \frac{t(4b-9)}{2(bt+3)(3t+2)} \geq 0.$$

(b) Since $f'(t) \leq 0$ for $0 \leq t \leq \frac{1}{\sqrt{b}}$ and $f'(t) \geq 0$ for $t \geq \frac{1}{\sqrt{b}}$, $f(t)$ is decreasing on

$\left[0, \frac{1}{\sqrt{b}}\right]$ and increasing on $\left[\frac{1}{\sqrt{b}}, \infty\right)$, hence $f(t) \geq f\left(\frac{1}{\sqrt{b}}\right)$.

(c) Since $t \leq b^2 \leq \frac{1}{\sqrt{b}}$, we have $f'(t) \leq 0$, $f(t)$ is decreasing on $[0, b^2]$, hence

$$f(t) \geq f(b^2).$$

Remark. Note that $\frac{2}{3}$ is the largest positive value of k such that

$$\frac{1}{ab+k} + \frac{1}{bc+k} + \frac{1}{cd+k} + \frac{1}{de+k} + \frac{1}{ea+k} \geq \frac{5}{1+k}$$

for any positive numbers a, b, c, d, e with $a \geq b \geq c \geq d \geq e$ and $abcde = 1$. For

$a = b = c = d := x$ and $e = \frac{1}{x^4}$, where $x \geq 1$, the constraints are satisfied, while the

inequality becomes

$$\frac{3}{x^2+k} + \frac{2x^3}{kx^3+1} \geq \frac{5}{1+k}.$$

For $x \rightarrow \infty$, we get the necessary condition $k \leq \frac{2}{3}$.

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SP.620 Let a, b, c be nonnegative real numbers such that at most one of them is larger than 1 and $ab + bc + ca = 3$. Prove that:

$$8 \left(\sqrt{1 + 3a^2} + \sqrt{1 + 3b^2} + \sqrt{1 + 3c^2} \right) \leq 21 + 3(a + b + c)^2.$$

Proposed by Vasile Cîrtoaje – Romania

Solution by proposer

Assume that $a \geq 1 \geq b \geq c$ and write the inequality as $E \leq 0$, where

$$E = 8 \left(\sqrt{1 + 3a^2} + \sqrt{1 + 3b^2} + \sqrt{1 + 3c^2} \right) - 21 - 3(a + b + c)^2.$$

If $a = c$, then $a = b = c = 1$ and $E = 0$. Assume further $a > c$ and denote $p = a + b + c$.

For fixed p , consider that a, c and E are functions of b . By differentiating the equality

constraints $a + b + c = p$ and $ab + bc + ca = 3$, we get

$$a' + c' + 1 = 0, \quad (p - 1)a' + (p - c)c' + p - b = 0,$$

$$\text{hence } a'(b) = \frac{-(b-c)}{a-c}, \quad c'(b) = \frac{-(a-b)}{a-c}.$$

Since $a'(b) \leq 0$ and $c'(b) \leq 0$, the functions $a(b)$ and $c(b)$ are decreasing. We have

$$\frac{E'(b)}{24} = \frac{aa'}{\sqrt{1 + 3a^2}} + \frac{cc'}{\sqrt{1 + 3c^2}} + \frac{b}{\sqrt{1 + 3b^2}},$$

therefore

$$\begin{aligned} \frac{-(a-c)E'(b)}{24} &= \frac{(b-c)a}{\sqrt{1 + 3a^2}} + \frac{(a-b)c}{\sqrt{1 + 3c^2}} - \frac{(a-c)b}{\sqrt{1 + 3b^2}} = \\ &= (b-c)f(a) + (a-b)f(c) - (a-c)f(b), \end{aligned}$$

where $f(x) = \frac{x}{\sqrt{1 + 3x^2}}$. Since $f''(x) = \frac{-9x}{(1 + 3x^2)^{\frac{5}{2}}} \leq 0$,

$f(x)$ is concave on $[0, \infty)$. By Jensen's inequality, we have

$$\begin{aligned} (b-c)f(a) + (a-b)f(c) &\leq [(b-c) + (a-b)]f\left(\frac{(b-c)a + (a-b)c}{(b-c) + (a-b)}\right) \\ &= (a-c)f(b), \end{aligned}$$

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hence $E'(b) \geq 0$. So, $E(b)$ is increasing and has the maximum value when b is maximum (or a or c are minimum), hence when $b = 1$ or $c = 0$. Thus, it suffices to consider these cases.

Case 1: $b = 1$. We need to show that $a + c + ac = 3$ implies

$$8\left(\sqrt{1+3a^2} + \sqrt{1+3c^2}\right) \leq 5 + 3(1+a+c)^2.$$

Let $x = \frac{a+c}{2}$. From $3 = 2x + ac \leq 2x + x^2$, we get $x \geq 1$. By squaring, the required inequality becomes

$$8\left(6x^2 - 3ac + 1 + \sqrt{9a^2c^2 - 6ac + 12x^2 + 1}\right) \leq (3x^2 + 3x + 2)^2,$$

$$32\sqrt{3x^2 - 6x + 4} \leq 9x^4 + 18x^3 - 27x^2 - 36x + 68.$$

Since

$$9x^4 + 18x^3 - 27x^2 - 36x + 68 = 9(x-1)^2(x^2 + 4x - 4) + 8(9x^2 - 18x + 13) \geq 8(9x^2 - 18x + 13),$$

it suffices to show that

$$4\sqrt{3x^2 - 6x + 4} \leq 9x^2 - 18x + 13.$$

Squaring again, the inequality becomes

$$(9x^2 - 18x + 13)^2 \geq 16(3x^2 - 6x + 4),$$

$$27x^4 - 108x^3 + 170x^2 - 124x + 35 \geq 0,$$

$$(x-1)^2(27x^2 - 54x + 35) \geq 0.$$

It is true because $27x^2 - 54x + 35 > 27x^2 - 54x + 27 = 27(x-1)^2 \geq 0$.

Case 2: $c = 0$. We need to show that $ab = 3$ implies

$$8\left(\sqrt{1+3a^2} + \sqrt{1+3b^2}\right) \leq 20 + 3(a+b)^2.$$

Replacing a and b with $a\sqrt{3}$ and $b\sqrt{3}$, respectively, we need to show that $ab = 1$ implies

$$8\left(\sqrt{1+9a^2} + \sqrt{1+9b^2}\right) \leq 20 + 9(a+b)^2.$$

Since

$$\sqrt{1+9a^2} + \sqrt{1+9b^2} \leq \sqrt{2(1+9a^2+1+9b^2)},$$

it suffices to show that

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$$8\sqrt{2 + 18a^2 + 18b^2} \leq 20 + 9(a + b)^2$$

Denoting $x = \frac{a+b}{2}$, the inequality becomes

$$2\sqrt{72x^2 - 34} \leq 9x^2 + 5, \quad 4(72x^2 - 34) \leq (9x^2 + 5)^2, \\ 81x^4 - 198x^2 + 161 \geq 0.$$

We have: $81x^4 - 198x^2 + 161 > 81x^4 - 198x^2 + 121 = (9x^2 - 11)^2 \geq 0$.

The proof is finished. The equality occurs for $a = b = c = 1$.

SP.621 Prove that $\frac{5}{4}$ is the least positive value of the power k such that the inequality $(ab)^k + (ac)^k + (ad)^k + (bc)^k + (bd)^k + (cd)^k \geq 6$ holds for any nonnegative real numbers a, b, c, d with $a \geq b \geq c \geq 1 \geq d$ and $ab + bc + cd + da = 4$.

Proposed by Vasile Cîrtoaje – Romania

Solution by proposer

For $a = b$ and $c = 1$, we have $(a + 1)(a + d) = 4$, and the inequality becomes $E \geq 0$, where

$$E = a^{2k} + 2a^k(1 + d^k) + d^k - 6$$

Assume that a and E are functions of d . If $d = 1$, then $a = 1$. We have

$$(2a + d + 1)a' + a + 1 = 0, \quad a'(1) = \frac{-1}{2},$$

$$(2a + d + 1)a'' + 2(a')^2 + 2a' = 0, \quad a''(1) = \frac{1}{8},$$

$$\frac{E'(d)}{k} = 2[a^{2k-1} + a^{k-1}(1 + d^k)]a' + 2a^k d^{k-1} + d^{k-1}, \quad E'(1) = 0,$$

$$\frac{E''(d)}{k} = 2[a^{2k-1} + a^{k-1}(1 + d^k)]a'' + 2[(2k-1)a^{2k-2} + (k-1)a^{k-2}(1 + d^k)](a')^2 \\ + 4ka^{k-1}a' + 2(k-1)a^k d^{k-2} + (k-1)d^{k-2},$$

hence

$$\frac{E''(1)}{k} = \frac{3}{4} + \frac{8k-6}{4} - 2k + 2(k-1) + k - 1 = \frac{3(4k-5)}{4}.$$

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Since $E(1) = E'(1) = 0$, the condition $E''(1) \geq 0$ is necessary to have $E(d) \geq 0$ for $d \in [0, 1]$. This condition implies $k \geq \frac{5}{4}$. So, it suffices to prove the required inequality for $k = \frac{5}{4}$. Replacing a, b, c, d with a^4, b^4, c^4, d^4 , respectively, we need to show that $F \geq 0$ for

$$(a^4 + c^4)(b^4 + d^4) = 4, \quad a \geq b \geq c \geq 1 \geq d \geq 0,$$

where

$$F(ab)^5 + (ac)^5 + (ad)^5 + (bc)^5 + (bd)^5 + (cd)^5 - 6.$$

For fixed a and b , assume that d and F are functions of c . Since

$$d^3(a^4 + c^4)d' + c^3(b^4 + d^4) = 0,$$

we have

$$\begin{aligned} \frac{F'(c)}{5} &= c^4(a^5 + b^5 + d^5) + d^4(a^5 + b^5 + c^5)d' = c^4(a^5 + b^5 + d^5) - \\ &\quad - \frac{c^3 d(a^5 + b^5 + c^5)(b^4 + d^4)}{a^4 + c^4}. \end{aligned}$$

We will show that $F'(c) \geq 0$, which is equivalent to $G \geq 0$, where

$$G = c(a^5 + b^5 + d^5)(a^4 + c^4) - d(a^5 + b^5 + c^5)(b^4 + d^4).$$

For fixed b, c and d , we have

$$G(a) = a^5[c(a^4 + c^4) - d(b^4 + d^4)] + (b^5 + d^5)c(a^4 + c^4) - (b^5 + c^5)d(b^4 + d^4).$$

Since $c(a^4 + c^4) - d(b^4 + d^4) \geq 0$, $G(a)$ is increasing and

$$\begin{aligned} G(a) &\geq G(b) = b^5[c(b^4 + c^4) - d(b^4 + d^4)] + \\ &\quad + (b^5 + d^5)c(b^4 + c^4) - (b^5 + c^5)d(b^4 + d^4) \\ &= 2(2b^5 + d^5)c(b^4 + c^4) - (2b^5 + c^5)d(b^4 + d^4) = b^4 A, \end{aligned}$$

where

$$A = 2b^5(c - d) + 2b(c^5 - d^5) - cd(c^4 - d^4).$$

Since

$$A \geq 2b(c^5 - d^5) - cd(c^4 - d^4) \geq c(c^5 - d^5) - cd(c^4 - d^4) = c^5(c - d) \geq 0,$$

we have $G(a) \geq 0$, hence $F'(c) \geq 0$, $F(c)$ is increasing and has the minimum value when c is minimum, hence when $c = 1$. So, it suffices to consider the case $c = 1$. We need to show that $H \geq 0$ for $a \geq b \geq 1 \geq d$ such that $(a^4 + 1)(b^4 + d^4) = 4$, where

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$$H = (ab)^5 + (bd)^5 + (da)^5 + a^5 + b^5 + d^5 - 6.$$

From $4 = (a^4 + 1)(b^4 + d^4) > a^4 b^4 \geq b^8$, we $b < 2$. For fixed d , assume that b and H are functions of a . Since

$$b^3(a^4 + 1)b' + a^3(b^4 + d^4) = 0,$$

we have

$$\begin{aligned} \frac{H'(a)}{5} &= a^4(b^5 + d^5 + 1) + b^4(a^5 + d^5 + 1)b' = a^4(b^5 + d^5 + 1) - \\ &\quad - \frac{a^3 b(a^5 + d^5 + 1)(b^4 + d^4)}{a^4 + 1}. \end{aligned}$$

We will show that $H'(a) \geq 0$, which is equivalent to $K \geq 0$, where

$$\begin{aligned} K &= a(a^4 + 1)(b^5 + d^5 + 1) - b(b^4 + d^4)(a^5 + d^5 + 1) \\ &= (d^5 + 1 - bd^4)a^5 + (b^5 + d^5 + 1)a - b(b^4 + d^4)(d^5 + 1). \end{aligned}$$

Since

$$d^5 + 1 - bd^4 \geq d^8 + 1 - bd^4 \geq 2d^4 - bd^4 = d^4(2 - b) \geq 0,$$

we have

$$\begin{aligned} K &\geq (d^5 + 1 - bd^4)b^5 + (b^5 + d^5 + 1)b - b(b^4 + d^4)(d^5 + 1) = \\ &= b(b^5 + d^5 + 1)(1 - d^4) \geq 0. \end{aligned}$$

So, $H'(a) \geq 0$, $H(a)$ is increasing and has the minimum value when a is minimum, hence

when $a = b$. Thus, it suffices to consider the case $a = b$, when we need to show that

$$(b^4 + 1)(b^4 + d^4) = 4 \text{ implies } b^{10} + 2(d^5 + 1)b^5 + d^5 \geq 6.$$

Write this inequality as $d^5 \geq \frac{6 - 2b^5 - b^{10}}{1 + 2b^5}$.

Consider the nontrivial case when $6 - 2b^5 - b^{10} \geq 0$, hence when

$$1 \leq b^5 \leq \sqrt{7} - 1 < \sqrt{3}. \text{ The inequality is true if } (d^4)^5 \geq \left(\frac{6 - 2b^5 - b^{10}}{1 + 2b^5} \right)^4,$$

that is

$$\left(\frac{4 - b^4 - b^8}{1 + b^4} \right)^5 \geq \left(\frac{6 - 2b^5 - b^{10}}{1 + 2b^5} \right)^4.$$

We will show that

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$$\left(\frac{4 - b^4 - b^8}{1 + b^4}\right)^5 \geq \left(\frac{101 + 18b^5 - 44b^{10}}{25(1 + 2b^5)}\right)^5 \geq \left(\frac{6 - 2b^5 - b^{10}}{1 + 2b^5}\right)^4.$$

The left inequality is true if

$$\frac{4 - b^4 - b^8}{1 + b^4} \geq \frac{101 + 18b^5 - 44b^{10}}{25(1 + 2b^5)},$$

which is equivalent to

$$44b^{14} - 50b^{13} + 44b^{10} - 68b^9 - 25b^8 + 182b^5 - 126b^4 - 1 \geq 0,$$

$$(b - 1)^3 \left(44b^{11} + 82b^{10} + 114b^9 + 140b^8 + 204b^7 + 238b^6 + 217b^5 + 141b^4 + \right. \\ \left. + 10b^3 + 6b^2 + 3b + 1 \right) \geq 0.$$

Denoting $t = b^5 \in [1, \sqrt{3})$, the right inequality becomes

$$\left(\frac{101 + 18t - 44t^2}{25}\right)^5 \geq (1 + 2t)(6 - 2t - t^2)^4.$$

Since $101 + 18t - 44t^2 > 0$, it is true if $g(t) \geq 0$, where

$$g(t) = 5 \ln(101 + 18t - 44t^2) - 5 \ln 25 - \ln(1 + 2t) - 4 \ln(6 - 2t - t^2).$$

From

$$\begin{aligned} \frac{g'(t)}{2} &= \frac{5(9 - 44t)}{101 + 18t - 44t^2} - \frac{1}{1 + 2t} + \frac{4(1 + t)}{6 - 2t - t^2} \\ &= \frac{4(11t^4 + 139t^3 - 294t^2 + 127t + 17)}{(101 + 18t - 44t^2)(1 + 2t)(6 - 2t - t^2)} \\ &= \frac{4(t - 1)^2(11t^2 + 161t + 17)}{(101 + 18t - 44t^2)(1 + 2t)(6 - 2t - t^2)} \geq 0, \end{aligned}$$

it follows that $g(t)$ is increasing, hence $g(t) \geq g(1) = 0$.

For $k = \frac{5}{4}$, the equality occurs when $a = b = c = d = 1$.

SP.622 Compute the value of the integral:

$$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos t}{\frac{1}{\pi t} + 1} dt$$

Proposed by José Luis Díaz – Barrero – Spain

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Solution 1 by proposer

To compute the value of the integral, we can use the substitution property for definite integrals (King's Property):

$$\int_a^b f(t) dt = \int_a^b f(a+b-t) dt$$

For our integral, $a = -\frac{\pi}{4}$ and $b = \frac{\pi}{4}$, so $a + b = 0$. Substituting t with $-t$ gives

$$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos(-t)}{\pi^{\frac{1}{-t}} + 1} (-dt) = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos t}{\pi^{-\frac{1}{t}} + 1} dt$$

Since $\pi^{-\frac{1}{t}} = \frac{1}{\pi^{\frac{1}{t}}}$, the denominator simplifies as follows

$$\frac{1}{\pi^{-\frac{1}{t}} + 1} = \frac{\pi^{\frac{1}{t}}}{\pi^{\frac{1}{t}} + 1}$$

Thus, the integral can be rewritten as

$$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos t \cdot \pi^{\frac{1}{t}}}{\pi^{\frac{1}{t}} + 1} dt$$

Adding our original expression for I and this new expression together yields

$$2I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos t}{\pi^{\frac{1}{t}} + 1} dt + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos t \cdot \pi^{\frac{1}{t}}}{\pi^{\frac{1}{t}} + 1} dt = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos t dt$$

Now, we evaluate this straightforward integral to obtain

$$2I = [\sin t]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \sqrt{2} \Rightarrow I = \frac{\sqrt{2}}{2}$$

Solution 2 by Angel Plaza-Spain

Let $f(t) = \frac{\cos t}{\pi^{\frac{1}{t}} + 1}$. Then

$$f(t) + f(-t) = \frac{\cos t}{\pi^{\frac{1}{t}} + 1} + \frac{\cos(-t)}{\pi^{\frac{1}{-t}} + 1} = \cos t \left(\frac{1}{\pi^{\frac{1}{t}} + 1} + \frac{\pi^{\frac{1}{t}}}{1 + \pi^{\frac{1}{t}}} \right) = \cos t$$

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$$I = \int_0^{\frac{\pi}{4}} \cos t \, dt = \sin t \Big|_0^{\frac{\pi}{4}} = \frac{\sqrt{2}}{2}.$$

Solution 3 by Marin Chirciu – Romania

$$\begin{aligned} I &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos t}{1 + \pi^{\frac{1}{t}}} dt \stackrel{t \rightarrow -1}{=} \int_{\frac{\pi}{4}}^{-\frac{\pi}{4}} \frac{\cos(-t)}{1 + \pi^{\frac{1}{-t}}} (-dt) = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos t}{1 + \pi^{-\frac{1}{t}}} dt = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos t}{1 + \frac{1}{\pi^{\frac{1}{t}}}} dt = \\ &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\pi^{\frac{1}{t}} \cos t}{1 + \pi^{\frac{1}{t}}} dt. \end{aligned}$$

We obtain:

$$\begin{aligned} I + I &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos t}{1 + \pi^{\frac{1}{t}}} dt + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\pi^{\frac{1}{t}} \cos t}{1 + \pi^{\frac{1}{t}}} dt = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{(1 + \pi^{\frac{1}{t}}) \cos t}{1 + \pi^{\frac{1}{t}}} dt = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos t \, dt = \sin t \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \\ &= \\ &= \sin \frac{\pi}{4} - \sin \frac{-\pi}{4} = 2 \sin \frac{\pi}{4} = 2 \cdot \frac{\sqrt{2}}{2} = \sqrt{2} \end{aligned}$$

From $2I = \sqrt{2} \Rightarrow I = \frac{\sqrt{2}}{2}$. We deduce that $I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos t}{1 + \pi^{\frac{1}{t}}} dt = \frac{\sqrt{2}}{2}$.

Remark: The problem can be developed.

Let be $\lambda > 0$ fixed. Compute the value of the integral

$$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos t}{1 + \lambda^{\frac{1}{t}}} dt.$$

Marin Chirciu

Solution

$$\begin{aligned} I &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos t}{1 + \lambda^{\frac{1}{t}}} dt \stackrel{t \rightarrow -t}{=} \int_{\frac{\pi}{4}}^{-\frac{\pi}{4}} \frac{\cos(-t)}{1 + \lambda^{\frac{1}{-t}}} (-dt) = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos t}{1 + \lambda^{-\frac{1}{t}}} dt = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos t}{1 + \frac{1}{\lambda^{\frac{1}{t}}}} dt = \\ &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\lambda^{\frac{1}{t}} \cos t}{1 + \lambda^{\frac{1}{t}}} dt. \end{aligned}$$

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We obtain :

$$\begin{aligned}
 I + I &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos t}{1 + \lambda^{\frac{1}{t}}} dt + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\lambda^{\frac{1}{t}} \cos t}{1 + \lambda^{\frac{1}{t}}} dt + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\left(1 + \lambda^{\frac{1}{t}}\right) \cos t}{1 + \lambda^{\frac{1}{t}}} dt = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos t dt = \sin t \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \\
 &= \\
 &= \sin \frac{\pi}{4} - \sin \frac{-\pi}{4} = 2 \sin \frac{\pi}{4} = 2 \cdot \frac{\sqrt{2}}{2} = \sqrt{2}.
 \end{aligned}$$

From $2I = \sqrt{2} \Rightarrow I = \frac{\sqrt{2}}{2}$. We deduce that $I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos t}{1 + \lambda^{\frac{1}{t}}} dt = \frac{\sqrt{2}}{2}$.

Note: For $\lambda = \pi$ we obtain Problem SP.622. from RMM Number 42 – Autumn 2026.

SP.622. Compute the value of the integral

$$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos t}{1 + \pi^{\frac{1}{t}}} dt$$

Jose Luis Diaz – Barrero – Spain.

SP.623 Compute the following limit:

$$\lim_{n \rightarrow +\infty} \frac{1}{n^2} \sum_{k=1}^n k \left(\frac{2k^2 + 3n^2}{3k^2 + 2n^2} \right)$$

Proposed by José Luis Díaz – Barrero – Spain

Solution 1 by proposer

To compute the limit, we can transform the sum into a Riemann sum and evaluate it using definite integral. Since

$$\frac{k}{n^2} \cdot \frac{2k^2 + 3n^2}{3k^2 + 2n^2} = \left(\frac{k}{n} \right) \cdot \frac{2 \left(\frac{k}{n} \right)^2 + 3}{3 \left(\frac{k}{n} \right)^2 + 2} \cdot \frac{1}{n}$$

then

$$\lim_{n \rightarrow +\infty} \sum_{k=1}^n \left[\left(\frac{k}{n} \right) \cdot \frac{2 \left(\frac{k}{n} \right)^2 + 3}{3 \left(\frac{k}{n} \right)^2 + 2} \cdot \frac{1}{n} \right]$$

This expression is precisely the Riemann sum for the function $f(x) = x \cdot \frac{2x^2+3}{3x^2+2}$ on the interval $[0, 1]$ with step size $\Delta x = \frac{1}{n}$.

Taking the limit as $n \rightarrow +\infty$ converts the sum into the integral of $f(x)$ on $[0, 1]$, and we have

$$\lim_{n \rightarrow +\infty} \sum_{k=1}^n \left[\left(\frac{k}{n} \right) \cdot \frac{2 \left(\frac{k}{n} \right)^2 + 3}{3 \left(\frac{k}{n} \right)^2 + 2} \cdot \frac{1}{n} \right] = \int_0^1 x \cdot \frac{2x^2 + 3}{3x^2 + 2} dx$$

Since

$$\int x \frac{2x^2 + 2}{3x^2 + 2} dx = \frac{1}{3}x^2 + \frac{5}{18} \ln(3x^2 + 2) + C$$

then

$$\int_0^1 x \cdot \frac{2x^2 + 3}{3x^2 + 2} dx = \left[\frac{1}{3}x^2 + \frac{5}{18} \ln(3x^2 + 2) + C \right]_0^1 = \frac{1}{3} + \frac{5}{18} \ln\left(\frac{5}{2}\right)$$

and

$$\lim_{n \rightarrow +\infty} \frac{1}{n^2} \sum_{k=1}^n k \left(\frac{2k^2 + 3n^2}{3k^2 + 2n^2} \right) = \frac{1}{3} + \frac{5}{18} \ln\left(\frac{5}{2}\right)$$

Solution 2 by Marin Chirciu-Romania

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{k=1}^n k \left(\frac{2k^2 + 3n^2}{3k^2 + 2n^2} \right) &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{k}{n} \left(\frac{2 \frac{k^2}{n^2} + 3}{3 \frac{k^2}{n^2} + 2} \right) = \int_0^1 x \frac{2x^2 + 3}{3x^2 + 2} dx = \\ &= \int_0^1 \frac{2x^3 + 3x}{3x^2 + 2} dx = \\ &= \frac{1}{3} \int_0^1 \frac{6x^3 + 9x}{3x^2 + 2} dx = \frac{1}{3} \int_0^1 \left(2x + \frac{5x}{3x^2 + 2} \right) dx = \frac{1}{3} \left(\int_0^1 2x dx + \int_0^1 \frac{5x}{3x^2 + 2} dx \right) = \\ &= \frac{1}{3} \left(\int_0^1 2x dx + \frac{5}{6} \int_0^1 \frac{6x}{3x^2 + 2} dx \right) = \frac{1}{3} \left(x^2 \Big|_0^1 + \frac{5}{6} \ln(3x^2 + 2) \Big|_0^1 \right) = \frac{1}{3} \left(1 + \frac{5}{6} \ln \frac{5}{2} \right). \end{aligned}$$

We deduce that $\lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{k=1}^n k \left(\frac{2k^2 + 3n^2}{3k^2 + 2n^2} \right) = \frac{1}{3} \left(1 + \frac{5}{6} \ln \frac{5}{2} \right)$.

Remark: The problem can be developed. Compute the following limit:

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{k=1}^n k \left(\frac{\lambda k^2 + (\lambda + 1)n^2}{(\lambda + 1)k^2 + \lambda n^2} \right), \lambda > 0.$$

Marin Chirciu

Solution:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{k=1}^n k \left(\frac{\lambda k^2 + (\lambda + 1)n^2}{(\lambda + 1)k^2 + \lambda n^2} \right) &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{k}{n} \left(\frac{\lambda \frac{k^2}{n^2} + \lambda + 1}{(\lambda + 1) \frac{k^2}{n^2} + \lambda} \right) = \\ &= \int_0^1 x \frac{\lambda x^2 + \lambda + 1}{(\lambda + 1)x^2 + \lambda} dx = \\ &= \int_0^1 \frac{\lambda x^3 + (\lambda + 1)x}{(\lambda + 1)x^2 + \lambda} dx = \frac{1}{\lambda + 1} \int_0^1 \frac{\lambda(\lambda + 1)x^3 + (\lambda + 1)^2 x}{(\lambda + 1)x^2 + \lambda} dx = \\ &= \frac{1}{\lambda + 1} \int_0^1 \left(\lambda x + \frac{(2\lambda + 1)x}{(\lambda + 1)x^2 + \lambda} \right) dx = \frac{1}{\lambda + 1} \left(\int_0^1 \lambda x dx + \int_0^1 \frac{(2\lambda + 1)x}{(\lambda + 1)x^2 + \lambda} dx \right) = \\ &= \frac{1}{\lambda + 1} \left(\int_0^1 \lambda x dx + \frac{2\lambda + 1}{2(\lambda + 1)} \int_0^1 \frac{2(\lambda + 1)x}{(\lambda + 1)x^2 + \lambda} dx \right) = \\ &= \frac{1}{\lambda + 1} \left(\frac{\lambda x^2}{2} \Big|_0^1 + \frac{2\lambda + 1}{2(\lambda + 1)} \ln((\lambda + 1)x^2 + \lambda) \Big|_0^1 \right) = \frac{1}{\lambda + 1} \left(\frac{\lambda}{2} + \frac{2\lambda + 1}{2(\lambda + 1)} \ln \frac{2\lambda + 1}{\lambda} \right). \end{aligned}$$

We deduce that $\lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{k=1}^n k \left(\frac{\lambda k^2 + (\lambda + 1)n^2}{(\lambda + 1)k^2 + \lambda n^2} \right) = \frac{1}{\lambda + 1} \left(\frac{\lambda}{2} + \frac{2\lambda + 1}{2(\lambda + 1)} \ln \frac{2\lambda + 1}{\lambda} \right).$

Note: For $\lambda = 2$ we obtain Problem SP.623 from RMM Number 42 – Autumn 2026.

SP.623. Compute the following limit $\lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{k=1}^n k \left(\frac{2k^2 + 3n^2}{3k^2 + 2n^2} \right).$

Jose Luis Diaz – Barrero – Spain

SP.624 Let ABC be a triangle. Prove that:

$$\left(\frac{1}{\sin^2 A} + \frac{1}{\sin^2 B} + \frac{1}{\sin^2 C} \right) \left(\sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} \right) \geq 3$$

Proposed by José Luis Díaz – Barrero – Spain

Solution 1 by proposer

In what follows, we find a lower bound for each term in the LHS.

1. The sum of the inverse squares of sines is bounded below by

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$$\frac{1}{\sin^2 A} + \frac{1}{\sin^2 B} + \frac{1}{\sin^2 C} \geq 4$$

Indeed, using the fundamental Pythagorean trigonometric identity

$\csc^2 \theta = 1 + \cot^2 \theta$ (where $\csc \theta = \frac{1}{\sin \theta}$), we can rewrite each term in the sum as

$$\frac{1}{\sin^2 A} = 1 + \cot^2 A$$

$$\frac{1}{\sin^2 B} = 1 + \cot^2 B$$

$$\frac{1}{\sin^2 C} = 1 + \cot^2 C$$

Summing these three equations together gives

$$\frac{1}{\sin^2 A} + \frac{1}{\sin^2 B} + \frac{1}{\sin^2 C} = 3 + (\cot^2 A + \cot^2 B + \cot^2 C)$$

It is a well-known geometric and algebraic identity for the angles of any triangle that the sum of the squared cotangents satisfies

$$\cot^2 A + \cot^2 B + \cot^2 C \geq 1$$

Equality holds if and only if $\cot A = \cot B = \cot C = \frac{1}{\sqrt{3}}$, which corresponds to an equilateral triangle where $A = B = C = 60^\circ$.

Thus, we have

$$\frac{1}{\sin^2 A} + \frac{1}{\sin^2 B} + \frac{1}{\sin^2 C} \geq 3 + 1 = 4$$

2. The second factor of the LHS satisfies

$$\sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} \geq \frac{3}{4}$$

Indeed,

1. For any triangle ABC with interior angles A, B, C , the sum of half – angles satisfies the well-known trigonometric identity

$$\sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} = 1 - 2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

2. It is also known the geometric property for triangle half-angles that the product of their sines is bounded above by $\frac{1}{8}$. That is,

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$$\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \leq \frac{1}{8}$$

(Equality holds precisely when $\frac{A}{2} = \frac{B}{2} = \frac{C}{2} = 30^\circ$, that is when $\triangle ABC$ is equilateral triangle).

3. Substituting this upper bound into our identity, since we are subtracting the product, the maximum product yields the minimum possible value for the sum

$$1 - 2 \left(\frac{1}{8} \right) = 1 - \frac{1}{4} = \frac{3}{4}$$

An alternative way to obtain the same bound is:

1. Using the half-angle power – reducing identity $\sin^2 \frac{\theta}{2} = \frac{1 - \cos \theta}{2}$, we rewrite the sum for all three angles:

$$\sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} = \frac{1 - \cos A}{2} + \frac{1 - \cos B}{2} + \frac{1 - \cos C}{2}$$

2. Combining the terms gives:

$$= \frac{3 - (\cos A + \cos B + \cos C)}{2}$$

3. For any triangle, the sum of the cosines of its interior angles has a maximum value of $\frac{3}{2}$. That is,

$$\cos A + \cos B + \cos C \leq \frac{3}{2}$$

4. To find the minimum value of the fraction, we substitute the maximum possible value for the subtracted term $\left(\frac{3}{2} \right)$ and we get

$$\frac{3 - \frac{3}{2}}{2} = \frac{3 - \frac{3}{2}}{2} = \frac{3}{4}$$

Multiplying our lower bounds together gives

$$\left(\frac{1}{\sin^2 A} + \frac{1}{\sin^2 B} + \frac{1}{\sin^2 C} \right) \left(\sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} \right) \geq 4 \times \frac{3}{4} = 3$$

Equality case occurs when $A = B = C = 60^\circ$. That is, when $\triangle ABC$ is equilateral.

Solution 2 by Marin Chirciu-Romania

$$1. \sum \frac{1}{\sin^2 A} = \frac{p^2(p^2+2r^2-8Rr)+r^2(4R+r)^2}{4p^2r^2} \stackrel{\text{Gerretsen}}{\geq} 4.$$

$$2. \sum \sin^2 \frac{A}{2} = 1 - \frac{r}{2R} \stackrel{\text{Euler}}{\geq} \frac{3}{4}.$$

$$LHS = \sum \frac{1}{\sin^2 A} \sum \sin^2 \frac{A}{2} \geq 4 \cdot \frac{3}{4} = 3 = RHS$$

Equality if and only if the triangle is equilateral.

Remark: The problem can be developed and strengthened.

In $\triangle ABC$ holds:

$$\frac{3R}{2r} \leq \sum \frac{1}{\sin^2 A} \sum \sin^2 \frac{A}{2} \leq \frac{R}{r} \left(\frac{R}{r} - \frac{1}{2} \right)$$

Marin Chirciu

Solution

$$1. \frac{2R}{r} \leq \sum \frac{1}{\sin^2 A} \leq \frac{R^2}{r^2}, \text{ see } \sum \frac{1}{\sin^2 A} = \frac{p^2(p^2+2r^2-8Rr)+r^2(4R+r)^2}{4p^2r^2} \text{ and the Gerretsen's inequality}$$

$$16Rr - 5r^2 \leq p^2 \leq 4R^2 + 4Rr + 3r^2.$$

$$2. \sum \sin^2 \frac{A}{2} = 1 - \frac{r}{2R} \stackrel{\text{Euler}}{\geq} \frac{3}{4}.$$

Right hand inequality.

$$\sum \frac{1}{\sin^2 A} \sum \sin^2 \frac{A}{2} \leq \frac{R^2}{r^2} \left(1 - \frac{r}{2R} \right) = \frac{R}{r} \left(\frac{R}{r} - \frac{1}{2} \right).$$

Equality holds if and only if the triangle is equilateral.

Right hand inequality.

$$\sum \frac{1}{\sin^2 A} \sum \sin^2 \frac{A}{2} \geq \frac{2R}{r} \cdot \frac{3}{4} = \frac{3R}{2r}.$$

Equality holds if and only if the triangle is equilateral.

Remark: Left hand inequality strengthen Problem SP.624 from RMM Number 42 – Autumn 2026. We can write the inequalities.

In $\triangle ABC$ holds

$$3 \leq \frac{3R}{2r} \leq \sum \frac{1}{\sin^2 A} \sum \sin^2 \frac{A}{2} \leq \frac{R}{r} \left(\frac{R}{r} - \frac{1}{2} \right)$$

SP.625 If $n \geq 3$ is an integer, then prove that $\sqrt{n+2} + \sqrt{n-2}$ is an irrational number.

Proposed by José Luis Díaz – Barrero – Spain

Solution by proposer

We argue by contradiction. Suppose that

$$\sqrt{n+2} + \sqrt{n-2} = \frac{p}{q}, \quad (p, q) = 1.$$

Inverting the relation gives

$$\frac{q}{p} = \frac{1}{\sqrt{n+2} + \sqrt{n-2}} = \frac{\sqrt{n+2} - \sqrt{n-2}}{4}$$

Hence we get the system of equations

$$\begin{aligned} \sqrt{n+2} + \sqrt{n-2} &= \frac{p}{q}, \\ \sqrt{n+2} - \sqrt{n-2} &= \frac{4q}{p}. \end{aligned}$$

Adding these equations and dividing by 2 gives $\sqrt{n+2} = \frac{4q^2 + p^2}{2pq}$.

Squaring and rearranging terms, we get $4np^2q^2 = 16q^4 + p^4$. Suppose now that n, p and q are all positive integers with p and q relatively prime. The relationship

$4np^2q^2 = 16q^4 + p^4$ shows that p^4 , and hence p , is divisible by 2. Letting $p = 2r$ we

obtain $nr^2q^2 = q^4 + r^4$ from which we obtain $n = \left(\frac{q^2}{r^2}\right) + \left(\frac{r^2}{q^2}\right)$ and this occurs only when

$\frac{q}{r} = \pm 1$ and $n = \pm 2$ and this contradicts that $n \geq 3$.

Alternatively, from $4np^2q^2 = 16q^4 + p^4$ we get $p^4 = q^2(4np^2 - 16q^2)$ and

$\left(\frac{p^2}{q}\right)^2 = 4np^2 - 16q^2$ and q divides p^2 which is not possible because $q > 1$. This

contradicts the assumption that p and q are coprime. Thus, the number $\sqrt{n+2} + \sqrt{n-2}$ is irrational.

SP.626 If $a, b, c > 0, a + b + c = 1$ and $\lambda \geq 1$ then:

$$\sum \frac{a + \sqrt{bc}}{a + \lambda} \leq \frac{1 + 3\lambda}{4} + \frac{1}{\lambda + 1} \sum \sqrt{bc}$$

Proposed by Marin Chirciu – Romania

Solution 1 by proposer

We have:

$$\sum \frac{a + \sqrt{bc}}{a + \lambda} \leq \frac{1 + 3\lambda}{4} + \frac{1}{\lambda + 1} \sum \sqrt{bc} \Leftrightarrow \sum \left(\frac{a + \sqrt{bc}}{a + \lambda} - \frac{1}{\lambda + 1} \sqrt{bc} \right) \leq \frac{1 + 3\lambda}{4}$$

which follows from:

$$\begin{aligned} \sum \left(\frac{a + \sqrt{bc}}{a + \lambda} - \frac{1}{\lambda + 1} \sqrt{bc} \right) &= \sum \frac{(\lambda + 1)a + (1 - a)\sqrt{bc}}{2(a + 1)} \stackrel{AGM}{\leq} \\ &\leq \sum \frac{(\lambda + 1)a + (1 - a)\frac{b + c}{2}}{2(a + 1)} = \\ &= \sum \frac{2(\lambda + 1)a + (1 - a)(b + c)}{4(a + \lambda)} = \sum \frac{2(\lambda + 1)a + (1 - a)(1 - a)}{4(a + \lambda)} \\ &= \sum \frac{a^2 + 2\lambda a + 1}{4(a + \lambda)} \\ &= \sum \frac{a^2 + 2\lambda a + \lambda^2 + 1 - \lambda^2}{4(a + \lambda)} = \sum \frac{a^2 + 2\lambda a + \lambda^2}{4(a + \lambda)} + \sum \frac{1 - \lambda^2}{4(a + \lambda)} = \\ &= \sum \frac{(a + \lambda)^2}{4(a + \lambda)} + \sum \frac{1 - \lambda^2}{4(a + \lambda)} = \\ &= \sum \frac{a + \lambda}{4} + \sum \frac{1 - \lambda^2}{4(a + \lambda)} = \frac{\sum a + 3\lambda}{4} + \sum \frac{1 - \lambda^2}{4(a + \lambda)} = \\ &= \frac{1 + 3\lambda}{4} + \sum \frac{1 - \lambda^2}{4(a + \lambda)} \stackrel{\lambda \geq 1}{\geq} \frac{1 + 3\lambda}{4} \end{aligned}$$

Equality holds if and only if $a = b = c = \frac{1}{3}$ and $\lambda = 1$.

Note: For $\lambda = 1$ we obtain Cyclic Inequality proposed by Phan Ngoc Chau in RMM 2/2022.

If $a, b, c > 0, a + b + c = 1$ then:

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$$\sum \frac{a + \sqrt{bc}}{a + 1} \leq 1 + \frac{1}{2} \sum \sqrt{bc}$$

Phan Ngoc Chau – Vietnam

Solution 2 by Jose Luis Diaz Barrero-Spain

Let $S = \sum_{cyc} \sqrt{bc}$. Since $1 - a = b + c$, then so

$$\frac{a + \sqrt{bc}}{a + \lambda} = \frac{a}{a + \lambda} + \frac{\sqrt{bc}}{\lambda + 1} + \frac{\sqrt{bc}(b + c)}{(a + \lambda)(\lambda + 1)}$$

Multiplying by $\sqrt{bc}$ the inequality

$$\frac{1}{a + \lambda} = \frac{1}{\lambda + 1} + \frac{1 - a}{(a + \lambda)(\lambda + 1)},$$

yields

$$\frac{\sqrt{bc}}{a + \lambda} = \frac{\sqrt{bc}}{\lambda + 1} + \frac{\sqrt{bc}(1 - a)}{(a + \lambda)(\lambda + 1)}.$$

Using the constraint $a + b + c = 1$, or $1 - a = b + c$, we have

$$\frac{\sqrt{bc}}{a + \lambda} = \frac{\sqrt{bc}}{\lambda + 1} + \frac{\sqrt{bc}(b + c)}{(a + \lambda)(\lambda + 1)}.$$

Adding $\frac{a}{a + \lambda}$ to both sides, we get

$$\frac{a + \sqrt{bc}}{a + \lambda} = \frac{a}{a + \lambda} + \frac{\sqrt{bc}}{a + \lambda} = \frac{a}{a + \lambda} + \frac{\sqrt{bc}}{\lambda + 1} + \frac{\sqrt{bc}(b + c)}{(a + \lambda)(\lambda + 1)},$$

which is the desired identity.

Summing cyclically, the middle term contributes $\frac{S}{\lambda + 1}$, which cancels with the same term on the right-hand side of the claim. Thus it suffices to prove

$$\sum_{cyc} \frac{a}{a + \lambda} + \frac{1}{\lambda + 1} \sum_{cyc} \frac{\sqrt{bc}(b + c)}{a + \lambda} \leq \frac{1 + 3\lambda}{4} \quad (1)$$

By AM-GM, $\sqrt{bc} \leq \frac{b + c}{2}$, hence $\sqrt{bc}(b + c) \leq \frac{(1 - a)^2}{2}$, so it is enough to show

$$\sum_{cyc} g(a) \leq \frac{1 + 3\lambda}{4}, \text{ where } g(a) = \frac{a}{a + \lambda} + \frac{(1 - a)^2}{2(\lambda + 1)(a + \lambda)}.$$

Combining over a common denominator, the numerator simplifies to

$$2(\lambda + 1)a + (1 - a)^2 = a^2 + 2\lambda a + 1 = (a + \lambda)^2 + (1 - \lambda^2),$$

so

$$g(a) = \frac{a + \lambda}{2(\lambda + 1)} + \frac{1 - \lambda}{2(a + \lambda)}.$$

Summing and using $a + b + c = 1$, we obtain

$$\sum_{cyc} g(a) = \frac{1 + 3\lambda}{2(\lambda + 1)} + \frac{1 - \lambda}{2} \sum_{cyc} \frac{1}{a + \lambda}.$$

Since $\lambda \geq 1$, we have $1 - \lambda \leq 0$ while $\sum_{cyc} \frac{1}{a + \lambda} > 0$, so

$$\sum_{cyc} g(a) \leq \frac{1 + 3\lambda}{2(\lambda + 1)}.$$

Moreover $\lambda \geq 1 \Rightarrow 2(\lambda + 1) \geq 4$, and $1 + 3\lambda > 0$, so

$$\frac{1 + 3\lambda}{2(\lambda + 1)} \leq \frac{1 + 3\lambda}{4}.$$

Combining the last two displays gives $\sum_{cyc} g(a) \leq \frac{1 + 3\lambda}{4}$, which proves (1) and hence the

claim. Equality holds iff $\lambda = 1$ and $a = b = c = \frac{1}{3}$.

SP.627 Solve for reals:

$$\sqrt[3]{(1+x)^2} + \sqrt[3]{(1-x)^2} = \sqrt[3]{1+x} + \sqrt[3]{1-x}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution 1 by Daniel Sitaru – Romania

Denote: $\begin{cases} a = \sqrt[3]{1+x} \\ b = \sqrt[3]{1-x} \end{cases} \Rightarrow a^3 + b^3 = 2$

Denote: $\begin{cases} S = a + b \\ P = ab \end{cases} \Rightarrow \begin{cases} S^3 - 3SP = 2 \\ S^2 - 2P = S \end{cases}$

$$2P = S^2 - S \Rightarrow P = \frac{S^2 - S}{2}$$

$$S^3 - 3S \cdot \frac{S^2 - S}{2} = 2 \Rightarrow 2S^3 - 3S^3 + 3S^2 = 4$$

$$-S^3 + 3S^2 - 4 = 0 \Rightarrow S^3 - 3S^2 + 4 = 0$$

$$S^3 + S^2 - 4S^2 + 4 = 0 \Rightarrow S^2(S + 1) - 4(S^2 - 1) = 0$$

$$S^2(S + 1) - 4(S - 1)(S + 1) = 0 \Rightarrow (S + 1)(S^2 - 4S + 4) = 0$$

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$$S + 1 = 0 \Rightarrow S = -1 \Rightarrow P = 1 \Rightarrow z^2 - Sz + P = 0 \Rightarrow z^2 + z + 1 = 0; \Delta < 0$$

$$S^2 - 4S + 4 = 0 \Rightarrow (S - 2)^2 = 0 \Rightarrow S = 2 \Rightarrow P = 1$$

$$\Rightarrow a = b = 1 \Rightarrow \sqrt[3]{1-x} = \sqrt[3]{1+x} = 1 \Rightarrow x = 0$$

Solution 2 by Marin Chirciu – Romania

With the substitution $(\sqrt[3]{1+x}, \sqrt[3]{1-x}) = (a, b) \Rightarrow \begin{cases} a^3 + b^3 = 2 \\ a^2 + b^2 = a + b \end{cases} \Rightarrow$

$$\Rightarrow \begin{cases} S^3 - 3SP = 2 \\ S^2 - 2P = S \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} S^3 - 3SP = 2 \\ P = \frac{S^2 - S}{2} \end{cases} \Leftrightarrow \begin{cases} S^3 - 3S \cdot \frac{S^2 - S}{2} = 2 \\ P = \frac{S^2 - S}{2} \end{cases} \Leftrightarrow \begin{cases} S^3 - 3S^2 - 4 = 0 \\ P = \frac{S^2 - S}{2} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} (S - 2)^2(S + 1) = 0 \\ P = \frac{S^2 - S}{2} \end{cases} \Leftrightarrow (S, P) \in \{(2, 1), (-1, 1)\}.$$

$$1. (S, P) = (2, 1) \Leftrightarrow (a, b) = (1, 1) \Leftrightarrow (\sqrt[3]{1+x}, \sqrt[3]{1-x}) = (1, 1) \Leftrightarrow x = 0.$$

$$2. (S, P) = (-1, 1) \Leftrightarrow a^2 - Sa + P = 0 \Leftrightarrow a^2 + a + 1 = 0, \Delta = -3 < 0, a \notin \mathbb{R}$$

We deduce that $x = 0$ is the unique solution of the equation.

SP.628 If $a, b, c > 0$; $(a + b + c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) = 17$ then:

$$(a^2 + b^2 + c^2) \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right) \geq 73$$

Proposed by Daniel Sitaru – Romania

Solution by proposer

Denote: $u = \frac{a}{b} + \frac{b}{c} + \frac{c}{a}$; $v = \frac{b}{a} + \frac{c}{b} + \frac{a}{c}$

$$17 = (a + b + c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) = \left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \right) + \left(\frac{b}{a} + \frac{a}{c} + \frac{c}{b} \right) + 3 = u + v + 3$$

$$\Rightarrow 17 = u + v + 3 \Rightarrow u + v = 14$$

$$(a^2 + b^2 + c^2) \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right) = \left(\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} \right) + \left(\frac{b^2}{a^2} + \frac{a^2}{c^2} + \frac{c^2}{b^2} \right) + 3 =$$

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$$= 3 + (u^2 - 2v) + (v^2 - 2u) = 3 + \left(\frac{u^2}{1} + \frac{v^2}{1}\right) - 2(u + v) \geq$$

$$\stackrel{\text{BERGSTROM}}{\geq} 3 + \frac{(u + v)^2}{1 + 1} - 2(u + v) = 3 + \frac{14^2}{2} - 2 \cdot 14 = 3 + 98 - 28 = 73$$

Equality holds for:

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = \frac{b}{a} + \frac{a}{c} + \frac{c}{b} = 7$$

SP.629 If $a, b > 0$; $a + b \leq 2$ then:

$$\left(\frac{2+a}{1+a} + \frac{1-2b}{1+2b}\right) \left(\frac{2+3a}{1+3a} + \frac{1-6b}{1+6b}\right) > \frac{64}{105}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$\begin{aligned} \frac{2+a}{1+a} + \frac{1-2b}{1+2b} &= \left(\frac{1}{1+a} + 1\right) + \left(\frac{2}{1+2b} - 1\right) = \\ &= \frac{1}{1+a} + \frac{2}{1+2b} = \frac{2}{2+2a} + \frac{2}{1+2b} = \\ &= 2 \left(\frac{1}{2+2a} + \frac{1}{1+2b}\right) \stackrel{\text{BERGSTROM}}{\geq} 2 \left(\frac{(1+1)^2}{2+2a+1+2b}\right) = \\ &= 2 \cdot \frac{4}{3+4b} = \frac{8}{3+2(a+b)} \geq \frac{8}{3+2 \cdot 2} = \frac{8}{7} \end{aligned}$$

$$\frac{2+a}{1+a} + \frac{1-2b}{1+2b} \geq \frac{8}{7} \quad (1)$$

$$\begin{aligned} \frac{2+3a}{1+3a} + \frac{1-6b}{1+6b} &= \left(\frac{1}{1+3a} + 1\right) + \left(\frac{2}{1+6b} - 1\right) = \\ &= \frac{1}{1+3a} + \frac{2}{1+6b} = \frac{2}{2+6a} + \frac{2}{1+6b} = 2 \left(\frac{1}{2+6a} + \frac{1}{1+6b}\right) \geq \\ &\stackrel{\text{BERGSTROM}}{\geq} 2 \cdot \frac{(1+1)^2}{2+6a+1+6b} = \frac{8}{3+6(a+b)} \geq \frac{8}{3+6 \cdot 2} = \frac{8}{15} \end{aligned}$$

$$\frac{2+3a}{1+a} + \frac{1-6b}{1+6b} \geq \frac{8}{15} \quad (2)$$

By multiplying (1);(2):

$$\left(\frac{2+a}{1+a} + \frac{1-2b}{1+2b}\right) \left(\frac{2+3a}{1+3a} + \frac{1-6b}{1+6b}\right) > \frac{8}{7} \cdot \frac{8}{15} = \frac{64}{105}$$

Solution 2 by Jose Luis Diaz Barrero-Spain

If $a, b > 0$ and $a + b \leq 2$, then prove that

$$\left(\frac{2+a}{1+a} + \frac{1-2b}{1+2b}\right) \left(\frac{2+3a}{1+3a} + \frac{1-6b}{1+6b}\right) \geq \frac{64}{105}$$

$$\frac{2+a}{1+a} = 1 + \frac{1}{1+a}, \quad \frac{1-2b}{1+2b} = -1 + \frac{2}{1+2b},$$

the first bracket becomes

$$\frac{2+a}{1+a} + \frac{1-2b}{1+2b} = \frac{1}{1+a} + \frac{2}{1+2b}.$$

Likewise,

$$\frac{2+3a}{1+3a} + \frac{1-6b}{1+6b} = \frac{1}{1+3a} + \frac{2}{1+6b}.$$

So it suffices to prove, for $a, b > 0$ with $a + b \leq 2$,

$$F(a, b) := \left(\frac{1}{1+a} + \frac{2}{1+2b}\right) \left(\frac{1}{1+3a} + \frac{2}{1+6b}\right) \geq \frac{64}{105}.$$

Fix $a > 0$. Each bracket of $F(a, b)$ is a positive constant plus a decreasing function of b ,

hence each bracket is a positive, decreasing function of b . A product of two positive

decreasing functions is decreasing, so $b \rightarrow F(a, b)$ is decreasing.

Since the constraint gives $0 < b \leq 2 - a$, decreasingness yields

$F(a, b) \geq F(a, 2 - a)$ for all admissible a, b .

So it suffices to prove $F(a, 2 - a) \geq \frac{64}{105}$ for $a \in (0, 2)$.

Set $b = 2 - a$. Combine each bracket into a single fraction

$$\frac{1}{1+a} + \frac{2}{5-2a} = \frac{(5-2a) + 2(1+a)}{(1+a)(5-2a)} = \frac{7}{(1+a)(5-2a)},$$

since $(5-2a) + 2(1+a) = 7$ identically. Similarly,

$$\frac{1}{1+3a} + \frac{2}{13-6a} = \frac{(13-6a) + 2(1+3a)}{(1+3a)(13-6a)} = \frac{15}{(1+3a)(13-6a)}.$$

$$F(a, 2-a) = \frac{7 \cdot 15}{(1+a)(5-2a)(1+3a)(13-6a)} = \frac{105}{A(a)B(a)},$$

where

$$A(a) = (1+a)(5-2a), \quad B(a) = (1+3a)(13-6a).$$

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Both $A(a), B(a) > 0$ for $a \in (0, 2)$, since $A > 0$ on $(-1, \frac{5}{2})$ and $B > 0$ on $(-\frac{1}{3}, \frac{13}{6})$.

Next, we bound $A(a)$ and $B(a)$ via AM-GM:

- **Bounding A :** Note $2(1 + a) + (5 - 2a) = 7$ is constant. By AM-GM,

$$2(1 + a)(5 - 2a) \leq \left(\frac{2(1 + a) + (5 - 2a)}{2} \right)^2 = \left(\frac{7}{2} \right)^2 = \frac{49}{4} \Rightarrow A(a) \leq \frac{49}{8},$$

with equality when $2(1 + a) = 5 - 2a$, i.e. $a = \frac{3}{4}$.

- **Bounding B :** Note $2(1 + 3a) + (13 - 6a) = 15$ is constant. By AM-GM,

$$2(1 + 3a)(13 - 6a) \leq \left(\frac{2(1 + 3a) + (13 - 6a)}{2} \right)^2 = \left(\frac{15}{2} \right)^2 = \frac{225}{4} \Rightarrow B(a) \leq \frac{225}{8},$$

with equality when $2(1 + 3a) = 13 - 6a$, i.e. $a = \frac{11}{12}$.

Since $A(a), B(a) > 0$, multiplying the two bounds gives

$$A(a)B(a) \leq \frac{49}{8} \cdot \frac{225}{8} = \frac{11025}{64} = \frac{105^2}{64}.$$

$$F(a, 2 - a) = \frac{105}{A(a)B(a)} \geq \frac{105}{\frac{11025}{64}} = \frac{105 \cdot 64}{105^2} = \frac{64}{105}.$$

Combining with the preceding, yields

$$F(a, b) \geq F(a, 2 - a) \geq \frac{64}{105} \text{ for all } a, b > 0, a + b \leq 2.$$

This is precisely the desired inequality. The two AM-GM steps attain equality at $a = \frac{3}{4}$ and

$a = \frac{11}{12}$ respectively, which cannot hold simultaneously, so the bound $\frac{64}{105}$ is never actually

attained. The true minimum of the left-hand side is slightly larger (numerically about

0.6132, attained near $a \approx 0.86, b \approx 1.14$). Thus $\frac{64}{105}$ is a clean, provable, though not

sharp, lower bound.

SP.630 In $\triangle ABC$ the following relationship holds:

$$\frac{a^2}{h_b^3} + \frac{b^2}{h_c^3} + \frac{c^2}{h_a^3} \geq \frac{8}{3R}$$

Proposed by Nguyen Hung Cuong – Vietnam

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Solution 1 by Daniel Sitaru – Romania

$$\begin{aligned}
 \frac{a^2}{h_b^3} + \frac{b^2}{h_c^3} + \frac{c^2}{h_a^3} &\geq 3 \sqrt[3]{\frac{(abc)^2}{(h_a h_b h_c)^3}} = \\
 &= \frac{3}{h_a h_b h_c} \cdot \sqrt[3]{(4Rrs)^2} \stackrel{\text{EULER}}{\geq} \frac{3}{\frac{2F}{a} \cdot \frac{2F}{b} \cdot \frac{2F}{c}} \cdot \sqrt[3]{(4 \cdot 2r \cdot rs)^2} = \\
 &= \frac{3abc}{8F^3} \cdot \sqrt[3]{(8r^2 s)^2} \stackrel{\text{MITRINOVIC}}{\geq} \frac{3 \cdot 4RF}{8F^3} \cdot \sqrt[3]{(8r^2 \cdot 3\sqrt{3}r)^2} = \frac{3R}{2F^2} \cdot \sqrt[3]{64 \cdot 27 \cdot r^6} = \\
 &= \frac{3R}{2r^2 s^2} \cdot 4 \cdot 3 \cdot r^2 = \frac{9 \cdot 2 \cdot R}{s^2} \stackrel{\text{MITRINOVIC}}{\geq} \frac{18R}{\frac{27R^2}{4}} = \frac{72}{27R} = \frac{8}{3R}
 \end{aligned}$$

Equality holds for $a = b = c$.

Solution 2 by Marin Chirciu – Romania

$$\begin{aligned}
 LHS = \sum \frac{a^2}{h_b^3} &\stackrel{\text{AM-GM}}{\geq} 3 \sqrt[3]{\prod \frac{a^2}{h_b^3}} = 3 \sqrt[3]{\frac{a^2 b^2 c^2}{h_a^3 h_b^3 h_c^3}} = 3 \sqrt[3]{\frac{16R^2 r^2 p^2}{\left(\frac{2r^2 p^2}{R}\right)^3}} = 3 \sqrt[3]{\frac{2R^5}{r^4 p^4}} \stackrel{\text{Mitrinovic}}{\geq} \\
 &\geq 3 \sqrt[3]{\frac{2R^5}{r^4 \left(\frac{27R^2}{4}\right)^2}} = \frac{2}{3r} \sqrt[3]{\frac{4R}{r}} \stackrel{\text{Euler}}{\geq} \frac{8}{3R} = RHS.
 \end{aligned}$$

We have used above $\frac{2}{3r} \sqrt[3]{\frac{4R}{r}} \geq \frac{8}{3R} \Leftrightarrow R^4 \geq 16r^4$, see $R \geq 2r$, (Euler).

Equality holds if and only if the triangle is equilateral.

Remark: The problem can be developed. In $\triangle ABC$ holds:

$$\sum \frac{a^n}{h_b^{n+1}} \geq \frac{2^{n+1}}{3^{\frac{n}{2}} R}, n \in \mathbb{N}$$

Marin Chirciu

Solution:

$$LHS = \sum \frac{a^n}{h_b^{n+1}} \stackrel{\text{AM-GM}}{\geq} 3 \sqrt[3]{\prod \frac{a^n}{h_b^{n+1}}} = 3 \sqrt[3]{\frac{a^n b^n c^n}{h_a^{n+1} h_b^{n+1} h_c^{n+1}}} = 3 \sqrt[3]{\frac{4^n R^n r^n p^n}{\left(\frac{2r^2 p^2}{R}\right)^{n+1}}} =$$

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$$= 3 \sqrt[3]{\frac{2^{n-1}R^{2n+1}}{r^{n+2}p^{n+2}}} \stackrel{\text{Mitrinovic}}{\geq} 3 \sqrt[3]{\frac{2^{n-1}R^{2n+1}}{r^{n+2}\left(\frac{3R\sqrt{3}}{2}\right)^{n+2}}} = \frac{1}{3^{\frac{n}{2}}} \sqrt[3]{\frac{2^{2n+1}R^{n-1}}{r^{n+2}}} \stackrel{\text{Euler}}{\geq} \frac{2^{n+1}}{3^{\frac{n}{2}}R} = \text{RHS}.$$

We have used above $\frac{1}{3^{\frac{n}{2}}} \sqrt[3]{\frac{2^{2n+1}R^{n-1}}{r^{n+2}}} \geq \frac{2^{n+1}}{3^{\frac{n}{2}}R} \Leftrightarrow R^{n+2} \geq (2r)^{n+2}$, see $R \geq 2r$, (Euler).

Equality holds if and only if the triangle is equilateral.

Note: For $n = 2$ we obtain Problem SP.630 from RMM Number 42 – Autumn 2026.

SP.630. In $\triangle ABC$ holds:

$$\sum \frac{a^2}{h_b^3} \geq \frac{8}{3R}$$

Nguyen Hung Cuong – Vietnam

Solution 3 by José Luis Díaz – Barrero – Spain

By the Law of Sines, we have $a = 2R \sin A$, $b = 2R \sin B$, $c = 2R \sin C$. Since

$S = \frac{1}{2}ac \sin B$ and $h_b = \frac{2S}{b}$, substituting gives

$$h_a = 2R \sin B \sin C, \quad h_b = 2R \sin A \sin C, \quad h_c = 2R \sin A \sin B$$

Then,

$$\frac{a^2}{h_b^3} = \frac{1}{2R \sin A \sin^3 C}, \quad \frac{b^2}{h_c^3} = \frac{1}{2R \sin^3 A \sin B}, \quad \frac{c^2}{h_a^3} = \frac{1}{2R \sin^3 B \sin C}$$

Writing $x = \sin A$, $y = \sin B$, $z = \sin C$, the inequality is equivalent to

$$\frac{1}{xz^3} + \frac{1}{x^3y} + \frac{1}{y^3z} \geq \frac{16}{3}.$$

On account of AM-GM inequality, we have

$$\frac{1}{xz^3} + \frac{1}{x^3y} + \frac{1}{y^3z} \geq \left(\frac{1}{xz^3} \cdot \frac{1}{x^3y} \cdot \frac{1}{y^3z} \right)^{\frac{1}{3}} = \frac{3}{(xyz)^4}.$$

Since $f(t) = \ln(\sin t)$ is concave on $(0, \pi)$, Jensen's inequality gives

$$\sin A \sin B \sin C \leq \sin^3 \left(\frac{A+B+C}{3} \right) = \sin^3 \frac{\pi}{3} = \frac{3\sqrt{3}}{8},$$

so

$$(xyz)^{\frac{4}{3}} \leq \left(\frac{3\sqrt{3}}{8}\right)^{\frac{4}{3}} = \frac{3^2}{8^{\frac{4}{3}}} = \frac{9}{16} \Rightarrow \frac{3}{(xyz)^{\frac{4}{3}}} \geq \frac{3}{\frac{9}{16}} = \frac{16}{3}$$

Finally, combining the preceding, yields

$$\frac{a^2}{h_b^3} + \frac{b^2}{h_c^3} + \frac{c^2}{h_a^3} = \frac{1}{2R} \left(\frac{1}{xz^3} + \frac{1}{x^3y} + \frac{1}{y^3z} \right) \geq \frac{1}{2R} \cdot \frac{16}{3} = \frac{8}{3R}.$$

Equality holds if and only if $A = B = C = \frac{\pi}{3}$. That is, when $\triangle ABC$ is equilateral.

UNDERGRADUATE PROBLEMS

UP.616 Calculate the following integral:

$$\int_0^{\infty} \frac{\ln x}{(x+1)(x^4+x^2+1)} dx$$

Proposed by Vasile Mircea Popa – Romania

Solution 1 by proposer

Let us denote:

$$I = \int_0^{\infty} \frac{\ln x}{(x+1)(x^4+x^2+1)} dx$$

$$A = \int_0^1 \frac{\ln x}{(x+1)(x^4+x^2+1)} dx = \int_0^1 \frac{(1-x) \ln x}{1-x^6} dx$$

$$B = \int_1^{\infty} \frac{\ln x}{(x+1)(x^4+x^2+1)} dx = \int_1^{\infty} \frac{(1-x) \ln x}{1-x^6} dx$$

We consider the integral A .

We have:

$$A = \int_0^1 \frac{\ln x}{1-x^6} dx - \int_0^1 \frac{x \ln x}{1-x^6} dx$$

We will calculate:

$$A_1 = \int_0^1 \frac{\ln x}{1-x^6} dx$$

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We make the variable change: $x^6 = y$; $x = y^{\frac{1}{6}}$.

We obtain:

$$A_1 = \frac{1}{36} \int_0^1 \frac{y^{\frac{5}{6}} \ln y}{1-y} dy$$

We will use the following known relationship:

$\int_0^1 \frac{z^a \ln z}{1-z} dz = -\Psi(a+1)$, $a \in \mathbb{R} \setminus \{-1, -2, \dots\}$, where $\Psi_1(x)$ is the trigamma function

We obtain:

$$A_1 = -\frac{1}{36} \Psi_1\left(\frac{1}{6}\right)$$

We will calculate:

$$A_2 = \int_0^1 \frac{x \ln x}{1-x^6} dx$$

Proceeding similarly, we obtain:

$$A_2 = -\frac{1}{36} \Psi\left(\frac{2}{6}\right)$$

We obtained the value of the integral A :

$$A = A_1 - A_2 = \frac{1}{36} \left[-\Psi_1\left(\frac{1}{6}\right) + \Psi\left(\frac{2}{6}\right) \right]$$

We consider the integral B . We make the variable change: $x = \frac{1}{t}$.

We obtain:

$$B = \int_0^1 \frac{(t-1)t^3 \ln t}{1-t^6} dt$$

By proceeding similarly to the integral A , we obtain:

$$B = \frac{1}{36} \left[\Psi_1\left(\frac{4}{6}\right) - \Psi_1\left(\frac{5}{6}\right) \right]$$

Result:

$$I = A + B$$

$$I = \frac{1}{36} \left[-\Psi_1\left(\frac{1}{6}\right) + \Psi\left(\frac{2}{6}\right) + \Psi_1\left(\frac{4}{6}\right) - \Psi_1\left(\frac{5}{6}\right) \right]$$

We use the reflection formula:

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$$\Psi_{1(x)} + \Psi_1(1-x) = \frac{\pi^2}{\sin^2(\pi x)}$$

We obtain:

$$\Psi_1\left(\frac{1}{6}\right) + \Psi_1\left(\frac{5}{6}\right) = \frac{\pi^2}{\sin^2 \frac{\pi}{6}} = 4\pi^2$$

$$\Psi\left(\frac{2}{6}\right) + \Psi_1\left(\frac{4}{6}\right) = \frac{\pi^2}{\sin^2 \frac{\pi}{3}} = \frac{4\pi^2}{3}$$

Result:

$$I = \frac{1}{36} \left[-4\pi^2 + \frac{4\pi^2}{3} \right], \quad I = -\frac{2\pi^2}{27}$$

Solution 2 by Jose Luis Diaz Barrero-Spain

Putting $x = \frac{1}{t}$, we have $dx = -\frac{dt}{t^2}$, $\ln x = -\ln t$, $(x+1)(x^4+x^2+1) = \frac{(1+t)(1+t^2+t^4)}{t^5}$, and

$$I = - \int_0^\infty \frac{t^3 \ln t}{(t+1)(t^4+t^2+1)} dt.$$

Define

$$A = \int_0^\infty \frac{\ln x}{(x+1)(x^4+x^2+1)} dx = I, \quad B = \int_0^\infty \frac{x^3 \ln x}{(x+1)(x^4+x^2+1)} dx,$$

such that $I = -B$. Adding up the two representations, yields

$$A - B = 2I = \int_0^\infty \frac{(1-x^3) \ln x}{(x+1)(x^4+x^2+1)} dx$$

Using $1-x^3 = (1-x)(1+x+x^2)$ and the factorization of x^4+x^2+1 , we have

$$\begin{aligned} \frac{1-x^3}{(x+1)(x^4+x^2+1)} &= \frac{(1-x)(x^2+x+1)}{(x+1)(x^2+x+1)(x^2-x+1)} \\ &= \frac{1-x}{(x+1)(x^2-x+1)} = \frac{1-x}{x^3+1}. \end{aligned}$$

Therefore,

$$2I = \int_0^\infty \frac{(1-x) \ln x}{x^3+1} dx = \int_0^\infty \frac{\ln x}{x^3+1} dx - \int_0^\infty \frac{x \ln x}{x^3+1} dx.$$

Using the well-known standard integral

$$\int_0^\infty \frac{x^{s-1}}{1+x^n} dx = \frac{\pi}{n \sin\left(\frac{\pi s}{n}\right)}, \quad 0 < s < n,$$

and computing its derivative respect to s yields

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$$\int_0^{\infty} \frac{x^{s-1} \ln x}{1+x^n} dx = -\frac{\pi^2}{n^2} \frac{\cos\left(\frac{\pi s}{n}\right)}{\sin^2\left(\frac{\pi s}{n}\right)}.$$

In particular, for Para $n = 3, s = 1$, we obtain

$$\int_0^{\infty} \frac{\ln x}{1+x^3} dx = -\frac{\pi^2}{9} \cdot \frac{\cos\left(\frac{\pi}{3}\right)}{\sin^2\left(\frac{\pi}{3}\right)} = -\frac{\pi^2}{9} \cdot \frac{\frac{1}{2}}{\frac{3}{4}} = -\frac{2\pi^2}{27}.$$

For $n = 3, s = 2$, we obtain

$$\int_0^{\infty} \frac{x \ln x}{1+x^3} dx = -\frac{\pi^2}{9} \cdot \frac{\cos\left(\frac{2\pi}{3}\right)}{\sin^2\left(\frac{2\pi}{3}\right)} = -\frac{\pi^2}{9} \cdot \frac{-\frac{1}{2}}{\frac{3}{4}} = \frac{2\pi^2}{27}$$

Then,

$$2I = \left(-\frac{2\pi^2}{27}\right) - \left(\frac{2\pi^2}{27}\right) = -\frac{4\pi^2}{27}.$$

and

$$I = \int_0^{\infty} \frac{\ln x}{(x+1)(x^4+x^2+1)} dx = -\frac{2\pi^2}{27}$$

UP.617 Calculate the integral:

$$\int_{-1}^1 \frac{x^2 \arccos(x)}{\sqrt{1-x^6}} dx$$

Proposed by Vasile Mircea Popa – Romania

Solution 1 by proposer

Let us denote:

$$A = \int_{-1}^1 \frac{x^2 \arccos(x)}{\sqrt{1-x^6}} dx; B = \int_{-1}^1 \frac{x^2 \arcsin(x)}{\sqrt{1-x^6}} dx$$

We have:

$$A + B = \frac{\pi}{2} \int_{-1}^1 \frac{x^2}{\sqrt{1-x^2}} dx$$

But:

$$B = \int_{-1}^1 \frac{x^2 \arcsin(x)}{\sqrt{1-x^6}} dx = 0$$

because the function to be integrate is odd.

Let us denote:

$$C = \int_{-1}^1 \frac{x^2}{\sqrt{1-x^6}} dx$$

We have:

$C = 2D$, where: $D = \int_0^1 \frac{x^2}{\sqrt{1-x^6}} dx$, because the function to be integrated is even.

We obtain: $A = \pi D$. We consider the integral D .

$$D = \int_0^1 \frac{x^2}{\sqrt{1-x^6}} dx$$

In the integral D we make the variable change $x^6 = y$

We obtain: $D = \frac{1}{6} \int_0^1 y^{-\frac{1}{2}} (1-y)^{-\frac{1}{2}} dy$. Let us denote:

$$E = \int_0^1 y^{-\frac{1}{2}} (1-y)^{-\frac{1}{2}} dy$$

So:

$$D = \frac{1}{6} E$$

We will use the Euler's Beta function:

$$B(p, q) = \int_0^1 t^{p-1} (1-t)^{q-1} dt$$

Result:

$$E = B\left(\frac{1}{2}, \frac{1}{2}\right), \quad E = \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} = \pi$$

Where $\Gamma(\alpha)$ is the Euler's Gamma function

$$D = \frac{1}{6} E = \frac{1}{6} \pi$$

$$A = \pi D = \frac{\pi^2}{6}$$

We obtained the value of the integral A :

$$A = \frac{\pi^2}{6}$$

Solution 2 by Jose Luis Diaz Barrero-Spain

$$g(x) = \frac{x^2}{\sqrt{1-x^6}},$$

which is an even function on $[-1, 1]$. Split the integral and substitute $x \rightarrow -x$ in the piece over $[-1, 0]$, to obtain

$$\begin{aligned} I &= \int_0^1 g(x) \arccos(-x) dx + \int_0^1 g(x) \arccos(x) dx = \\ &= \int_0^1 g(x) [\arccos(x) + \arccos(-x)] dx. \end{aligned}$$

Using the identity: $\arccos(x) + \arccos(-x) = \pi$, $x \in [-1, 1]$, this simplifies to

$$I = \pi \int_0^1 g(x) dx = \pi \int_0^1 \frac{x^2}{\sqrt{1-x^6}} dx.$$

Let $u = x^3$, so $du = 3x^2 dx$. When $x = 0$, $u = 0$ and when $x = 1$, $u = 1$. Then

$$\int_0^1 \frac{x^2}{\sqrt{1-x^6}} dx = \frac{1}{3} \int_0^1 \frac{du}{\sqrt{1-u^2}} = \frac{1}{3} [\arcsin(u)]_0^1 = \frac{1}{3} \cdot \frac{\pi}{2} = \frac{\pi}{6},$$

and $I = \pi \cdot \frac{\pi}{6} = \frac{\pi^2}{6}$.

UP.618 Find the curvilinear integral of the second kind:

$$\int_C (y+z) dx + (x+z) dy + (x+y) dz$$

$$C: \begin{cases} x(t) = t + 1 \\ y(t) = t + 2; t \in [0, 1] \\ z(t) = t + 3 \end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$P(x(t), y(t), z(t)) = y(t) + z(t) = t + 2 + t + 3 = 2t + 5$$

$$Q(x(t), y(t), z(t)) = x(t) + z(t) = t + 1 + t + 3 = 2t + 4$$

$$R(x(t), y(t), z(t)) = x(t) + y(t) = t + 1 + t + 2 = 2t + 3$$

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$$\begin{aligned} & \int_C (y+z) dx + (x+z) dy + (x+y) dz = \\ &= \int_0^1 ((2t+5) \cdot 1 + (2t+4) \cdot 1 + (2t+3) \cdot 1) dt = \\ &= \int_0^1 (6t+13) dt = 6 \cdot \frac{t^2}{2} \Big|_0^1 + 12t \Big|_0^1 = 3 + 12 = 15 \end{aligned}$$

Solution 2 by Jose Luis Diaz Barrero-Spain

$$\int_C (y+z) dx + (x+z) dy + (x+y) dz$$

where C is given by the parametric equations:

$$\begin{cases} x(t) = t + 1 \\ y(t) = t + 2 \\ z(t) = t + 3 \end{cases} \text{ for } t \in [0, 1]$$

1. Computing the differentials, we have $dx = dy = dz = dt$.

2. Substituting the parametric equations into the integral components, we get

$$y + z = (t + 2) + (t + 3) = 2t + 5$$

$$x + z = (t + 1) + (t + 3) = 2t + 4$$

$$x + y = (t + 1) + (t + 2) = 2t + 3$$

3. Summing the components and integrating from 0 to 1 yields

$$\begin{aligned} \int_C (y+z) dx + (x+z) dy + (x+y) dz &= \int_0^1 [(2t+5) + (2t+4) + (2t+3)] dt \\ &= \int_0^1 (6t+12) dt = [3t^2 + 12t]_0^1 = 15 \end{aligned}$$

UP.619 Find the curvilinear integral of the second kind:

$$\int_C yz dx + xz dy + xy dz$$

$$C: \begin{cases} x(t) = t \\ y(t) = 2t; t \in [0, 1] \\ z(t) = 3t \end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$P(x(t); y(t); z(t)) = y(t)z(t) = 2t \cdot 3t = 6t^2$$

$$Q(x(t); y(t); z(t)) = x(t)z(t) = t \cdot 3t = 3t^2$$

$$R(x(t); y(t); z(t)) = x(t)y(t) = t \cdot 2t = 2t^2$$

$$x'(t) = 1; y'(t) = 2; z'(t) = 3$$

$$\begin{aligned} \int_C yz \, dx + xz \, dy + xy \, dz &= \int_0^1 (6t^2 \cdot 1 + 3t^2 \cdot 2 + 2t^2 \cdot 3) \, dt = \\ &= 18 \int_0^1 t^2 \, dt = 18 \cdot \frac{t^3}{3} \Big|_0^1 = \frac{18}{3} = 6 \end{aligned}$$

Solution 2 by Jose Luis Diaz Barrero-Spain

1. Computing the differentials, we have $dx = dt, dy = 2dt, dz = 3dt$.

2. Substituting the parametric equations into the integral components, we get

$$yz \, dx = (2t)(3t) \, dt = 6t^2 \, dt$$

$$xz \, dy = (t)(3t)(2dt) = 6t^2 \, dt$$

$$xy \, dz = (t)(2t)(3dt) = 6t^2 \, dt$$

3. Adding up the components and integrating from 0 to 1 yields

$$\begin{aligned} \int_C yz \, dx + xz \, dy + xy \, dz &= \int_0^1 (6t^2 + 6t^2 + 6t^2) \, dt = \\ &= \int_0^1 18t^2 \, dt = 18 \cdot \frac{t^3}{3} \Big|_0^1 = \frac{18}{3} = 6 \end{aligned}$$

UP.620 Find the curvilinear integral of the first kind:

$$\Omega = \int_{\gamma} z \, ds; \gamma: \begin{cases} x = t \cos t \\ y = t \sin t; t \in [0, \pi] \\ z = \sqrt{3}t \end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution by proposer

$$x(t) = t \cos t \Rightarrow x'(t) = \cos t - t \sin t$$

$$y(t) = t \sin t \Rightarrow y'(t) = \sin t + t \cos t$$

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$$z(t) = \sqrt{3}t \Rightarrow z'(t) = \sqrt{3}$$

$$\begin{aligned} (x'(t))^2 + (y'(t))^2 + (z'(t))^2 &= (\cos t - t \sin t)^2 + (\sin t + t \cos t)^2 + (\sqrt{3})^2 = \\ &= \cos^2 t + t^2 \sin^2 t - 2t \sin t \cos t + \sin^2 t + t^2 \cos^2 t + 2t \sin t \cos t + 3 = \\ &= (\sin^2 t + \cos^2 t) + t^2(\sin^2 t + \cos^2 t) + 3 = t^2 + 4 \end{aligned}$$

$$ds = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt = \sqrt{t^2 + 4} dt$$

$$\begin{aligned} \Omega &= \int_{\gamma} z ds = \int_0^{\pi} \sqrt{3} t \cdot \sqrt{t^2 + 4} dt = \sqrt{3} \int_0^{\pi} t \cdot (t^2 + 4)^{\frac{1}{2}} dt = \\ &= \frac{\sqrt{3}}{2} \int_0^{\pi} (t^2 + 4)^{\frac{1}{2}} (t^2 + 4)^{\frac{1}{2}} dt = \frac{\sqrt{3}}{2} \cdot \frac{(t^2 + 4)^{\frac{3}{2}}}{\frac{3}{2}} \Big|_0^{\pi} = \\ &= \frac{\sqrt{3}}{3} \left((\pi^2 + 4)^{\frac{3}{2}} - 4^{\frac{3}{2}} \right) = \frac{\sqrt{3}}{3} \left((\pi^2 + 4) \sqrt{\pi^2 + 4} - 8 \right) \end{aligned}$$

UP.621 Find the curvilinear integral of the first kind:

$$\Omega = \int_{\gamma} (x + 2y + z) ds; \gamma = [AB]; \quad A(1, 2, 3); B(3, 4, 5)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$\begin{aligned} AB: \frac{x - x_A}{x_B - x_A} &= \frac{y - y_A}{y_B - y_A} = \frac{z - z_A}{z_B - z_A} \\ AB: \frac{x - 1}{3 - 1} &= \frac{y - 2}{4 - 2} = \frac{z - 3}{5 - 3} = t; t \in [0, 1] \\ AB: \begin{cases} x - 1 = 2t \\ y - 2 = 2t \\ z - 3 = 2t \end{cases} &\Rightarrow AB: \begin{cases} x = 1 + 2t \\ y = 2 + 2t \\ z = 3 + 2t \end{cases} \\ \begin{cases} x'(t) = (1 + 2t)' = 2 \\ y'(t) = (2 + 2t)' = 2 \\ z'(t) = (3 + 2t)' = 2 \end{cases} \end{aligned}$$

$$ds = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt = \sqrt{4 + 4 + 4} dt = 2\sqrt{3} dt$$

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$$\begin{aligned}\Omega &= \int_{\gamma} (x + 2y + z) \, ds = \\ &= \int_0^1 (1 + 2t + 2(2 + 2t) + 3 + 2t) \cdot 2\sqrt{3} \, dt = 2\sqrt{3} \int_0^1 (8 + 8t) \, dt = \\ &= 16\sqrt{3} \left(t \Big|_0^1 + \frac{t^2}{2} \Big|_0^1 \right) = 16\sqrt{3} \cdot \frac{3}{2} = 24\sqrt{3}\end{aligned}$$

Solution 2 by Jose Luis Diaz Barrero-Spain

To parametrize the segment from point $A(1, 2, 3)$ to point $B(3, 4, 5)$, we use the vector equation of a line

$$r(t) = A + t(B - A), \quad t \in [0, 1]$$

or in coordinates

$$x(t) = 1 + (3 - 1)t = 1 + 2t$$

$$y(t) = 2 + (4 - 2)t = 2 + 2t$$

$$z(t) = 3 + (5 - 3)t = 3 + 2t$$

Next, we find the derivatives of each coordinate with respect to t and compute ds . We have $x'(t) = 2, y'(t) = 2, z'(t) = 2$, and the arc length differential ds is given by

$$\begin{aligned}ds &= \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} \, dt \\ ds &= \sqrt{2^2 + 2^2 + 2^2} \, dt = \sqrt{4 + 4 + 4} \, dt = \sqrt{12} \, dt = 2\sqrt{3} \, dt\end{aligned}$$

Substituting the parameterizations and ds into the integral over the interval $t \in [0, 1]$ yields

$$\Omega = \int_0^1 [(1 + 2t) + 2(2 + 2t) + (3 + 2t)] \cdot 2\sqrt{3} \, dt = 16\sqrt{3} \int_0^1 (1 + t) \, dt$$

Now, integrate with respect to t to obtain

$$\Omega = 16\sqrt{3} \left[t + \frac{t^2}{2} \right]_0^1 = 16\sqrt{3} \left(1 + \frac{1}{2} \right) = 16\sqrt{3} \cdot \frac{3}{2} = 24\sqrt{3}$$

Thus,

$$\Omega = 24\sqrt{3}$$

UP.622 Find the surface integral of the first kind:

$$\Omega = \int \int_{\Sigma} (x + y + z) d\tau$$

$$\Sigma = \{(x, y, z) \in \mathbb{R}^3 | x = 2u + 3v, y = 3u + 2v, z = u + v; u, v \in [0, 1]\}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$x = \varphi(u, v) = 2u + 3v; y = \psi(u, v) = 3u + 2v, z = \chi(u, v) = u + v;$$

$$\frac{\partial \varphi}{\partial v} = 2; \frac{\partial \psi}{\partial u} = 3; \frac{\lambda \chi}{\lambda u} = 1$$

$$\frac{\lambda \varphi}{\lambda v} = 3; \frac{\partial \psi}{\partial v} = 2; \frac{\partial \chi}{\partial u} = 1$$

$$E = \left(\frac{\partial \varphi}{\partial u}\right)^2 + \left(\frac{\partial \psi}{\partial u}\right)^2 + \left(\frac{\partial \chi}{\partial u}\right)^2 = 2^2 + 3^2 + 1^2 = 14$$

$$G = \left(\frac{\partial \varphi}{\partial v}\right)^2 + \left(\frac{\partial \psi}{\partial v}\right)^2 + \left(\frac{\partial \chi}{\partial v}\right)^2 = 3^2 + 2^2 + 1^2 = 14$$

$$F = \left(\frac{\partial \varphi}{\partial u}\right)\left(\frac{\partial \varphi}{\partial v}\right) + \left(\frac{\partial \psi}{\partial u}\right)\left(\frac{\partial \psi}{\partial v}\right) + \left(\frac{\partial \chi}{\partial u}\right)\left(\frac{\partial \chi}{\partial v}\right) = 2 \cdot 3 + 3 \cdot 2 + 1 \cdot 1 = 13$$

$$d\tau = \sqrt{EG - F^2} du dv = \sqrt{14 \cdot 14 - 13^2} du dv = \sqrt{27} du dv$$

$$f(x, y, z) = x + y + z$$

$$f(\varphi(u, v), \psi(u, v), \chi(u, v)) = 2u + 3v + 3u + 2v + u + v = 6u + 6v$$

$$\begin{aligned} \Omega &= \int \int_{\Sigma} (x + y + z) d\tau = \int \int_D (6u + 6v) \sqrt{27} du dv = \\ &= 18\sqrt{3} \left(\int_0^1 u du \cdot \int_0^1 dv + \int_0^1 du \cdot \int_0^1 v dv \right) = 18\sqrt{3} \left(\frac{1}{2} \cdot 1 + 1 \cdot \frac{1}{2} \right) = 18\sqrt{3} \end{aligned}$$

Solution 2 by Jose Luis Diaz Barrero-Spain

We compute this integral carrying out the following steps:

- First, we compute the partial derivatives with respect to u and v : Let the position vector of the surface be $r(u, v) = (2u + 3v, 3u + 2v, u + v)$. Then,

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$$r_u = \frac{\partial r}{\partial u} = (2, 3, 1)$$

$$r_v = \frac{\partial r}{\partial v} = (3, 2, 1)$$

- Second, we compute the cross product $r_u \times r_v$:

$$r_u \times r_v = \begin{vmatrix} i & j & k \\ 2 & 3 & 1 \\ 3 & 2 & 1 \end{vmatrix} = 1i + 1j - 5k = (1, 1, -5)$$

- Third, we calculate the surface element differential $dS = |r_u \times r_v| du dv$:

$$|r_u \times r_v| = \sqrt{1^2 + 1^2 + (-5)^2} = \sqrt{1 + 1 + 25} = \sqrt{27} = 3\sqrt{3}$$

Thus, $dS = 3\sqrt{3} du dv$.

- Fourth, we substitute the parametrization into the integrand $x + y + z$ to obtain

$$x + y + z = (2u + 3v) + (3u + 2v) + (u + v) = 6(u + v).$$

- Fifth, we evaluate the double integral over $u, v \in [0, 1]$ and we get

$$\begin{aligned} \iint_S (x + y + z) dS &= \int_0^1 \int_0^1 (6u + 6v) \cdot 3\sqrt{3} du dv = \\ &= 3\sqrt{3} \int_0^1 \int_0^1 6(u + v) du dv = 18\sqrt{3} \int_0^1 \left[\frac{u^2}{2} + vu \right]_{u=0}^{u=1} dv = \\ &= 18\sqrt{3} \int_0^1 \left[\frac{u^2}{2} + vu \right]_{u=0}^{u=1} dv = 18\sqrt{3} \int_0^1 \left(\frac{1}{2} + v \right) dv = \\ &= 18\sqrt{3} \left[\frac{1}{2}v + \frac{v^2}{2} \right]_0^1 = 18\sqrt{3} \left(\frac{1}{2} + \frac{1}{2} \right) = 18\sqrt{3} \end{aligned}$$

UP.623 If $a, b, c > 0$, $a^2 + b^2 + c^2 = 1$ and $\lambda \geq \frac{1}{3}$ then:

$$\sum a^2 b \sum \frac{1}{\sqrt{a^2 + \lambda}} \leq \frac{3}{\sqrt{1 + 3\lambda}}$$

Proposed by Marin Chirciu – Romania

Solution by proposer

We prove: If $a, b, c > 0$, $a^2 + b^2 + c^2 = 1$ then:

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$$\sum a^2 b \leq \frac{1}{3}(a + b + c)$$

Proof: We have:

$$\begin{aligned} 3 \sum a^2 b &\stackrel{(1)}{\leq} \sum a^2 \sum a = 1 \cdot \sum a = \sum a, \text{ where } (1) \Leftrightarrow 3 \sum a^2 b \leq \sum a^2 \sum a \Leftrightarrow \\ &\Leftrightarrow \sum a(b - c)^2 \geq 0 \end{aligned}$$

Lemma 2: If $a, b, c > 0$, $a^2 + b^2 + c^2 = 1$ and $\lambda \geq \frac{1}{3}$ then

$$\left(\sum \frac{1}{\sqrt{a^2 + \lambda}} \right)^2 \leq 3 \left(3 - \frac{\lambda(a + b + c)^2}{1 + 3\lambda} \right).$$

Proof: We have

$$\begin{aligned} \left(\sum \frac{1}{\sqrt{a^2 + \lambda}} \right)^2 &\stackrel{CBS}{\leq} 3 \sum \frac{1}{a^2 + \lambda} = 3 \frac{1}{\lambda} \sum \left(1 - \frac{a^2}{a^2 + \lambda} \right) = \frac{3}{\lambda} \left(3 - \sum \frac{a^2}{a^2 + \lambda} \right) \stackrel{CS}{\leq} \\ &\leq \frac{3}{\lambda} \left(3 - \frac{(a + b + c)^2}{\sum(a^2 + \lambda)} \right) = \frac{3}{\lambda} \left(3 - \frac{(a + b + c)^2}{1 + 3\lambda} \right) \end{aligned}$$

Let's get back to the main problem. Using the Lemmas, we obtain:

$$\begin{aligned} LHS^2 &= \left(\sum a^2 b \sum \frac{1}{\sqrt{a^2 + \lambda}} \right)^2 \stackrel{Lemma 1+2}{\leq} \frac{1}{9}(a + b + c)^2 \cdot \frac{3}{\lambda} \left(3 - \frac{(a + b + c)^2}{1 + 3\lambda} \right) \stackrel{(2)}{\leq} \\ &\leq \frac{9}{1 + 3\lambda} = RHS^2, \end{aligned}$$

$$\text{where } (2) \Leftrightarrow \frac{1}{9}(a + b + c)^2 \cdot \frac{3}{\lambda} \left(3 - \frac{(a + b + c)^2}{1 + 3\lambda} \right) \leq \frac{9}{1 + 3\lambda} \stackrel{t = (a + b + c)^2}{\Leftrightarrow} \frac{1}{9}t \cdot \frac{3}{\lambda} \left(3 - \frac{t}{1 + 3\lambda} \right) \leq \frac{9}{1 + 3\lambda} \Leftrightarrow$$

$$t(3 + 9\lambda - t) \leq 27\lambda \Leftrightarrow t^2 - 3(3\lambda + 1)t + 27\lambda \geq 0 \Leftrightarrow (t - 3)(t - 9\lambda) \geq 0,$$

which follows from $t \leq 3$ and $\lambda \geq \frac{1}{3}$,

$$(\text{see } t = (a + b + c)^2 \leq 3(a^2 + b^2 + c^2) = 3 \cdot 1 = 3).$$

Equality holds if and only if $a = b = c = \frac{1}{\sqrt{3}}$.

UP.624 Solve for reals:

$$\sqrt[4]{1 + x} + \sqrt[4]{1 - x} - \sqrt[4]{1 - x^2} = 1$$

Proposed by Nguyen Hung Cuong – Vietnam

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Solution by Daniel Sitaru – Romania

$$\text{Denote: } \begin{cases} \sqrt[4]{1+x} = a \\ \sqrt[4]{1-x} = b \end{cases}; \begin{cases} S = a + b \\ P = ab > 0 \end{cases} \Rightarrow \begin{cases} a + b - ab = 1 \\ a^4 + b^4 = 2 \end{cases} \Rightarrow \begin{cases} S = 1 + P \\ (S^2 - 2P)^2 - 2P^2 = 2 \end{cases}$$

$$((1 + P)^2 - 2P)^2 - 2P^2 = 2$$

$$(1 + 2P + P^2 - 2P)^2 - 2P^2 = 2$$

$$(P^2 + 1)^2 - 2P^2 = 2 \Rightarrow P^4 + 2P^2 + 1 - 2P^2 = 2$$

$$\Rightarrow P^4 = 1 \Rightarrow P = 1 \quad (\text{because } P > 0)$$

$$P = 1 \Rightarrow S = 2 \Rightarrow \begin{cases} a = 1 \\ b = 1 \end{cases} \Rightarrow \begin{cases} \sqrt[4]{1+x} = 1 \\ \sqrt[4]{1-x} = 1 \end{cases} \Rightarrow x = 0$$

UP.625 Find:

$$\Omega = \iint_D (3x + 2y) \, dx \, dy$$

$$D = \text{Int}(\Delta AOB); 0(0, 0); A(2, 4); B(5, 4)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$OA: \begin{vmatrix} x & y & 1 \\ 0 & 0 & 1 \\ 2 & 4 & 1 \end{vmatrix} = 0; OA: 0 + 2y - 0 - 0 - 4x - 0 = 0$$

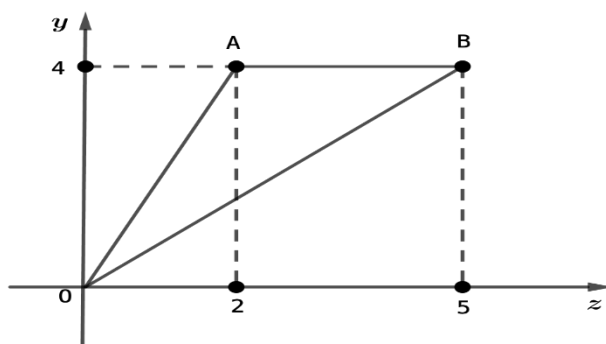
$$OA: 4x = 2y; OA: x = \frac{y}{2}$$

$$OB: \begin{vmatrix} x & y & 1 \\ 0 & 0 & 1 \\ 5 & 4 & 1 \end{vmatrix} = 0; OB: 0 + 5y - 0 - 0 - 4x - 0 = 0$$

$$OB: 5y = 4x; OB: x = \frac{5y}{4}$$

$$\begin{aligned} \Omega &= \int \int_D (3x + 2y) \, dx \, dy = \int_0^4 \left(\int_{\frac{y}{2}}^{\frac{5y}{4}} (3x + 2y) \, dx \right) dy = \int_0^4 \left(\frac{3x^2}{2} \Big|_{\frac{y}{2}}^{\frac{5y}{4}} + 2yx \Big|_{\frac{y}{2}}^{\frac{5y}{4}} \right) dy = \\ &= \int_0^4 \left(\frac{3}{2} \left(\frac{25y^2}{16} - \frac{y^2}{4} \right) + 2y \cdot \frac{3y}{4} \right) dy = \int_0^4 \left(\frac{3}{32} \cdot 21y^2 + \frac{48y^2}{32} \right) dy = \end{aligned}$$

$$= \frac{111}{32} \int_0^4 y^2 dy = \frac{111}{32} \cdot \frac{y^3}{3} \Big|_0^4 = \frac{111}{32} \cdot \frac{64}{3} = 74$$



Solution 2 by Jose Luis Diaz Barrero-Spain

Find:

$$\iint_D (3x + 2y) dx dy$$

where D is the interior of triangle AOB with vertices: $O(0, 0)$, $A(2, 4)$, $B(5, 4)$

To evaluate the double integral

$$\iint_D (3x + 2y) dx dy$$

where D is the interior of triangle AOB with vertices $O(0, 0)$, $A(2, 4)$, and $B(5, 4)$, we first determine the boundaries of the region. The triangle is bounded by three lines:

- Line OA : Passes through $(0, 0)$ and $(2, 4)$. Equation: $y = 2x \Rightarrow x = \frac{y}{2}$
- Line OB : Passes through $(0, 0)$ and $(5, 4)$. Equation: $y = \frac{4}{5}x \Rightarrow x = \frac{5}{4}y$
- Line AB : Horizontal line at $y = 4$. For a fixed y between 0 and 4, x ranges from the left boundary $x = \frac{y}{2}$ to the right boundary $x = \frac{5}{4}y$.

The integral becomes

$$\int_0^4 \int_{\frac{y}{2}}^{\frac{5y}{4}} (3x + 2y) dx dy$$

Integrating with respect to x yields

$$\int_{\frac{y}{2}}^{\frac{5y}{4}} (3x + 2y) dx = \left[\frac{3}{2}x^2 + 2yx \right]_{\frac{y}{2}}^{\frac{5y}{4}} = \left(\frac{3}{2} \left(\frac{5y}{4} \right)^2 + 2y \left(\frac{5y}{4} \right) \right) - \left(\frac{3}{2} \left(\frac{y}{2} \right)^2 + 2y \left(\frac{y}{2} \right) \right) = \frac{111}{32} y^2$$

Now, integrating with respect to y , we obtain

$$\int_0^4 \frac{111}{32} y^2 dy = \frac{111}{32} \left[\frac{y^3}{3} \right]_0^4 = \frac{111}{32} \left(\frac{64}{3} \right) = 74 \text{ and } \iint_D (3x + 2y) dx dy = 74.$$

UP.626 Find:

$$\Omega = \iint_D (2x + 3y) \, dx \, dy$$

$$D = \text{Int}(\triangle AOB); O(0, 0); A(2, 3); B(2, 5)$$

Proposed by Daniel Sitaru – Romania

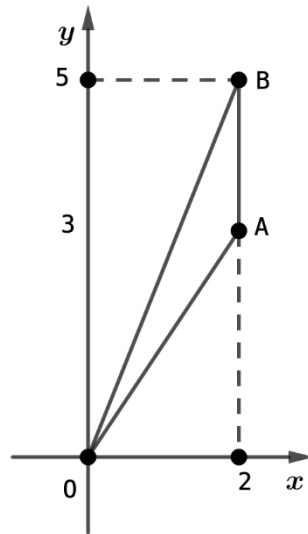
Solution 1 by proposer

$$OA: \begin{vmatrix} x & y & 1 \\ 0 & 0 & 1 \\ 2 & 3 & 1 \end{vmatrix} = 0; OA: 0 + 2y + 0 - 0 - 3x - 0 = 0$$

$$OA: 2y = 3x \Rightarrow y = \frac{3x}{2}$$

$$OB: \begin{vmatrix} x & y & 1 \\ 0 & 0 & 1 \\ 2 & 5 & 1 \end{vmatrix} = 0; OB: 0 + 2y - 0 - 0 - 5x - 0 = 0$$

$$OB: 2y = 5x \Rightarrow y = \frac{5x}{2}$$



$$\begin{aligned} \Omega &= \iint_D (2x + 3y) \, dx \, dy = \int_0^2 \left(\int_{\frac{3x}{2}}^{\frac{5x}{2}} (2x + 3y) \, dy \right) dx = \int_0^2 \left(2xy \Big|_{\frac{3x}{2}}^{\frac{5x}{2}} + 3 \cdot \frac{y^2}{2} \Big|_{\frac{3x}{2}}^{\frac{5x}{2}} \right) dx = \\ &= \int_0^2 \left(2x \left(\frac{5x}{2} - \frac{3x}{2} \right) + \frac{3}{2} \left(\frac{25x^2}{4} - \frac{9x^2}{4} \right) \right) dx = \int_0^2 \left(2x^2 + \frac{3}{2} \cdot \frac{16x^2}{4} \right) dx = \\ &= \int_0^2 (2x^2 + 6x^2) \, dx = 8 \cdot \frac{x^3}{3} \Big|_0^2 = 8 \left(\frac{2^3}{3} - \frac{0^3}{3} \right) = \frac{64}{3} \end{aligned}$$

Solution 2 by Jose Luis Diaz Barrero-Spain

To compute the integral

$$\int \int_D (2x + 3y) dx dy$$

where D is the interior of triangle AOB with vertices $O(0, 0)$, $A(2, 3)$, and $B(2, 5)$, we first determine the boundaries of the region. The triangle is bounded by three lines:

- Line OA : Passes through $(0, 0)$ and $(2, 3)$. Equation: $y = \frac{3}{2}x$.
- Line OB : Passes through $(0, 0)$ and $(2, 5)$. Equation: $y = \frac{5}{2}x$
- Line AB : Vertical Base AB : $x = 2$.

Thus,

$$\int \int_D (2x + 3y) dx dy = \int_0^2 \int_{\frac{3}{2}x}^{\frac{5}{2}x} (2x + 3y) dy dx$$

Integrating with respect to y yields

$$\int_{\frac{3}{2}x}^{\frac{5}{2}x} (2x + 3y) dy = \left[2xy + \frac{3}{2}y^2 \right]_{\frac{3}{2}x}^{\frac{5}{2}x} = 8x^2$$

Now, integrate with respect to x , we get

$$\int_0^2 8x^2 dx = 8 \left[\frac{x^3}{3} \right]_0^2 = 8 \left(\frac{8}{3} \right) = \frac{64}{3},$$

and

$$\int \int_D (2x + 3y) dx dy = \frac{64}{3}$$

UP.627 Find the surface integral of the second kind:

$$\iint_{\Sigma} 2x dydz + ydzdx + 5zdx dy$$

where Σ is the surface of the ellipsoid: $E: \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{25} = 1$

Proposed by Daniel Sitaru – Romania

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Solution 1 by proposer

$$P(x, y, z) = 2x, Q(x, y, z) = y, R(x, y, z) = 5z$$

$$\frac{\partial P}{\partial x} = 2; \frac{\partial Q}{\partial y} = 1; \frac{\partial R}{\partial z} = 5, \operatorname{div} \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 2 + 1 + 5 = 8$$

$$\operatorname{Vol}(E) = \frac{4\pi abc}{3}; a^4 =; b^2 = 9; c^2 = 25$$

$$\Rightarrow a = 2, b = 3, c = 5; \operatorname{Vol}(E) = \frac{4\pi \cdot 2 \cdot 3 \cdot 5}{3} = 40\pi$$

$$\begin{aligned} \iint_{\Sigma} 2x \, dx \, dy + y \, dz \, dx + 5z \, dx \, dy &= \iiint_{\Omega} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dx \, dy \, dz = \\ &= \iiint_{\Omega} \operatorname{div} \vec{F} \, dx \, dy \, dz = \iiint_{\Omega} 8 \, dx \, dy \, dz = 8 \iiint_{\Omega} dx \, dy \, dz = 8 \operatorname{Vol}(E) = 8 \cdot 40\pi = 320\pi \end{aligned}$$

Solution 2 by Jose Luis Diaz Barrero-Spain

To find the surface integral we will use the Gauss's Divergence Theorem. It relates a surface integral over a closed, oriented surface S to a triple integral over the solid region V enclosed by S . That is,

$$\int \int_S (P \, dy \, dz + Q \, dz \, dx + R \, dx \, dy) = \int \int \int_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dV$$

Here, our vector field components are $P = 2x$, $Q = y$ and $R = 5z$. Next, we find the divergence of the vector field $F = \langle 2x, y, 5z \rangle$. That is:

$$\nabla \cdot F = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(5z)$$

$$\nabla \cdot F = 2 + 1 + 5 = 8$$

The surface integral now simplifies to

$$8 \int \int \int_V dV$$

where $\int \int \int_V dV$ is simply the volume of the ellipsoid V .

Now, we calculate the Volume of the Ellipsoid. The standard equation for an ellipsoid is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

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From our given equation, the semi-axes are $a = 2, b = 3, c = 5$. It is well-known that the formula for the volume of an ellipsoid is $\text{Volume} = \frac{4}{3}\pi abc$. Substituting the values of a, b and c yields

$$\text{Volume} = \frac{4}{3}\pi(2)(3)(5) = 40\pi,$$

and the value of the surface integral is

$$\int \int_S 2x \, dy \, dz + y \, dz \, dx + 5z \, dx \, dy = 8 \times (40\pi) = 320\pi$$

UP.628 Find the general solution for the system:

$$\begin{cases} y'_1 = y_1 + y_2 + y_3 \\ y'_2 = 2y_2 + y_3 \\ y'_3 = 3y_3 \end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$\begin{pmatrix} y'_1 \\ y'_2 \\ y'_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \quad \det \left(\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) = 0$$

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0, \quad (1-\lambda)(2-\lambda)(3-\lambda) = 0 \begin{cases} \lambda_1 = 1 \\ \lambda_2 = 2 \\ \lambda_3 = 3 \end{cases}$$

For $\lambda = 1$

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} v_2 + v_3 \\ v_2 + v_3 \\ 2v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ \Rightarrow \begin{cases} v_2 + v_3 = 0 \\ v_3 = 0 \end{cases} \Rightarrow v_2 = 0 \Rightarrow \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \text{eigen vector}$$

For $\lambda = 2$

$$\begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -v_1 + v_2 + v_3 \\ v_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

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$$\Rightarrow \begin{cases} v_3 = 0 \\ -v_1 + v_2 + v_3 \end{cases} \Rightarrow v_1 = v_2 \Rightarrow \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \text{eigen vector}$$

For $\lambda = 3$

$$\begin{pmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -2v_1 + v_2 + v_3 \\ -v_2 + v_3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} -2v_1 + v_2 + v_3 = 0 \\ -v_2 + v_3 = 0 \end{cases} \Rightarrow \begin{cases} -2v_1 = -2v_2 \\ v_3 = v_2 \end{cases} \Rightarrow v_1 = v_2$$

$$\Rightarrow \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_1 \\ v_1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \text{eigen vector}$$

Solution:

$$\begin{pmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{pmatrix} = c_1 \cdot e^{1 \cdot x} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \cdot e^{2 \cdot x} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_3 \cdot e^{3 \cdot x} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{pmatrix} = \begin{pmatrix} c_1 e^x + c_2 e^{2x} + c_3 e^{3x} \\ c_2 e^{2x} + c_3 e^{3x} \\ c_3 e^{3x} \end{pmatrix} \Rightarrow \begin{cases} y_1(x) = c_1 e^x + c_2 e^{2x} + c_3 e^{3x} \\ y_2(x) = c_2 e^{2x} + c_3 e^{3x} \\ y_3(x) = c_3 e^{3x} \end{cases}$$

Solution 2 by Jose Luis Diaz Barrero-Spain

First, we write the system in matrix form $y' = Ay$ as

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

Since A is an upper triangular matrix, its eigenvalues are the elements on the main diagonal:

$$\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$$

The eigenvectors corresponding to each eigenvalue are

- For $\lambda_1 = 1$: Solving $(A - I)v_1 = 0$, we get the eigenvector $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.
- For $\lambda_2 = 2$: Solving $(A - 2I)v_2 = 0$, we get the eigenvector $v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

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- For $\lambda_3 = 3$: Solving $(A - 3I)v_3 = 0$, we get the eigenvector $v_3 = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$.

Combining the eigenvalues and eigenvectors, the general solutions is given by:

$$\begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix} = C_1 e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + C_3 e^{3t} \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$$

or in component form

$$\begin{cases} y_1(t) = C_1 e^t + C_2 e^{2t} + 3C_3 e^{3t} \\ y_2(t) = C_2 e^{2t} + 2C_3 e^{3t} \\ y_3(t) = 2C_3 e^{3t} \end{cases}$$

where C_1, C_2, C_3 are arbitrary constants.

UP.629 Find the general solution:

$$y'' - 5y' + 4y = xe^x$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

The homogenous equation: $y'' - 5y' + 4y = 0$

The characteristic equation: $r^2 - 5r + 4 = 0$

Solutions: $r_1 = 1; r_2 = 4$, $S = \{e^x; e^{4x}\}$ – the fundamental system of solutions

The general solution for homogenous equation:

$$y_h(x) = c_1 e^x + c_2 e^{4x}$$

$$\begin{cases} c_1'(x)e^x + c_2'(x)e^{4x} = 0 \\ c_1'(x)(e^x)' + c_2'(x)(e^{4x})' = xe^x \end{cases}$$

$$\begin{cases} c_1'(x)e^x + c_2'(x)e^{4x} = 0 \\ c_1'(x)e^x + 4c_2'(x)e^{4x} = xe^x \end{cases}$$

$$w(x) = \begin{vmatrix} e^x & e^{4x} \\ e^x & 4e^{4x} \end{vmatrix} = 4e^{5x} - e^{5x} = 3e^{5x}$$

$$c_1'(x) = \frac{\begin{vmatrix} 0 & e^{4x} \\ xe^x & 4e^{4x} \end{vmatrix}}{w(x)} = \frac{-xe^{5x}}{3e^{5x}} = -\frac{x}{3}$$

$$c_2'(x) = \frac{\begin{vmatrix} e^x & 0 \\ e^x & xe^x \end{vmatrix}}{w(x)} = \frac{xe^{2x}}{3e^{5x}} = \frac{x}{3e^{3x}} = \frac{xe^{-3x}}{3}$$

$$c_1'(x) = -\frac{x}{3}$$

$$c_1(x) = \int \left(-\frac{x}{3}\right) dx = -\frac{1}{3} \int x dx = -\frac{1}{3} \cdot \frac{x^2}{2} = -\frac{x^2}{6}$$

$$c_2'(x) = \frac{x}{3} e^{-3x}$$

$$\begin{aligned} c_2(x) &= \int \frac{x}{3} e^{-3x} dx = -\frac{1}{9} \int x(e^{-3x})' dx = -\frac{x}{9} e^{-3x} + \frac{1}{9} \int e^{-3x} dx = \\ &= -\frac{x}{9} e^{-3x} - \frac{1}{27} e^{-3x} \end{aligned}$$

$$y_p(x) = -\frac{x}{3} e^x + \left(-\frac{x}{3} e^{-3x} - \frac{1}{27} e^{-3x}\right) e^{4x}$$

$$y_p = -\frac{x}{3} e^x - \left(\frac{x}{3} + \frac{1}{27}\right) e^x$$

$$y_p = \left(-\frac{2x}{3} - \frac{1}{27}\right) e^x - \text{a particular solution}$$

The general solution:

$$y(x) = y_h(x) + y_p(x) = c_1 e^x + c_2 e^{4x} - \left(\frac{2x}{3} + \frac{1}{27}\right) e^x$$

Solution 2 by Jose Luis Diaz Barrero-Spain

The associated homogeneous equation is

$$y'' - 5y' + 4y = 0,$$

and its characteristic equation is:

$$r^2 - 5r + 4 = 0 \Rightarrow (r - 1)(r - 4) = 0$$

This gives roots $r_1 = 1$ and $r_2 = 4$, and the homogeneous solution is

$$y_h(x) = C_1 e^x + C_2 e^{4x}$$

To find the particular solution we will use the method of undetermined coefficients: The right-hand side is $f(x) = xe^x$. Since $r = 1$ is a single root of the characteristic equation, we will use the trial solution

$$y_p(x) = x(Ax + B)e^x = (Ax^2 + Bx)e^x$$

Next, we compute the derivatives of $y_p(x)$:

$$y_p' = (2Ax + B)e^x + (Ax^2 + Bx)e^x = [Ax^2 + (2A + B)x + B]e^x$$

$$\begin{aligned} y_p'' &= (2Ax + 2A + B)e^x + [Ax^2 + (2A + B)x + B]e^x \\ &= [Ax^2 + (4A + B)x + (2A + 2B)]e^x \end{aligned}$$

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Substituting into the original differential equation yields

$$y'' - 5y' + 4y = xe^x,$$

and factoring out e^x , we get

$$[Ax^2 + (4A + b)x + (2A + 2B)] - 5[Ax^2 + (2A + B)x + B] + 4[Ax^2 + Bx] = x$$

Group the terms by powers of x :

- For x^2 : $A - 5A + 4A = 0$ (matches correctly)
- For x : $(4A + B) - 5(2A + B) + 4B = -6A = 1 \Rightarrow A = -\frac{1}{6}$
- For the constant term: $(2A + 2B) - 5B = 2A - 3B = 0 \Rightarrow 3B = 2A \Rightarrow$
 $\Rightarrow B = \frac{2}{3}A = -\frac{1}{9}$

Thus, the particular solution is

$$y_p(x) = \left(-\frac{1}{6}x^2 - \frac{1}{9}x\right)e^x,$$

and the general solution is

$$y(x) = y_h(x) + y_p(x) = C_1e^x + C_2e^{4x} - \left(\frac{1}{6}x^2 + \frac{1}{9}x\right)e^x$$

UP.630 Find:

$$\Omega = \int \int_D x^2 dx dy$$

$$D = \{(x, y) | 0 \leq y \leq 9 - x^2; -3 \leq x \leq 3\}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$\begin{aligned} \Omega &= \int \int_D x^2 dx dy = \int_{-3}^3 \left(\int_0^{9-x^2} x^2 dy \right) dx = \int_{-3}^3 \left(x^2 y \Big|_0^{9-x^2} \right) dx = \\ &= \int_{-3}^3 (x^2(9 - x^2 - 0)) dx = \int_{-3}^3 (9x^2 - x^4) dx = \\ &= 9 \cdot \frac{x^3}{3} \Big|_{-3}^3 - \frac{x^5}{5} \Big|_{-3}^3 = 3(3^3 - (-3)^3) - \frac{1}{5}(3^5 - (-3)^5) = \\ &= 3 \cdot 2 \cdot 3^3 - \frac{1}{5} \cdot 2 \cdot 3^5 = 2 \cdot 81 - \frac{2 \cdot 243}{5} = \frac{810 - 486}{5} = \frac{324}{5} \end{aligned}$$

Solution 2 by Jose Luis Diaz Barrero

To evaluate the double integral, we set it up as in iterated integral

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$$\int_{-3}^3 \int_0^{9-x^2} x^2 dy dx$$

First, integrating with respect to y , we get

$$\int_0^{9-x^2} x^2 dy = x^2[y]_0^{9-x^2} = x^2(9-x^2) = 9x^2 - x^4$$

Now, we integrate the resulting expression with respect to x , and we obtain

$$\int_{-3}^3 (9x^2 - x^4) dx = \left[\frac{9x^3}{3} - \frac{x^5}{5} \right]_{-3}^3 = \left[3x^3 - \frac{x^5}{5} \right]_{-3}^3 = \frac{324}{5}$$