ROMANIAN MATHEMATICAL MAGAZINE

UP.615 Prove that $\frac{8}{5}$ is the smallest positive value of the constant k such that

$$\frac{1}{(a+b)^2+k}+\frac{1}{(b+c)^2+k}+\frac{1}{(c+a)^2+k}\le\frac{3}{4+k}$$

for all side lengths a,b,c of a triangle with $ab+bc+ca=3$.

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Solution 1 by proposer

For $\frac{a}{2}=b=c=\sqrt{\frac{3}{5}}$, the constraint is satisfied and the inequality becomes

$$\frac{10}{27+5k}+\frac{5}{12+5k}\le\frac{3}{4+k},$$

that is $k\ge\frac{8}{5}$. To show that $\frac{8}{5}$ is the smallest positive value of k, we need to prove that $E\le 0$ for $k=\frac{8}{5}$, where

$$E=\frac{1}{(a+b)^2+k}+\frac{1}{(b+c)^2+k}+\frac{1}{(c+a)^2+k}-\frac{3}{4+k}.$$

Assume $a\ge b\ge c$ and denote

$$X=(a+b)^2+k,\qquad Y=(a+c)^2+k,\qquad Z=(b+c)^2+k.$$

For fixed a, assume that c and E are functions of b. By differentiating the equality constraint, we get

$$(a+b)c'+a+c=0.$$

Therefore

$$\frac{E'(b)}{2}=-\frac{a+b}{X^2}-\frac{b+c}{Z^2}-\left[\frac{a+c}{Y^2}+\frac{b+c}{Z^2}\right]c'$$
$$=-\frac{a+b}{X^2}+\frac{b+c}{Z^2}+\left[\frac{a+c}{Y^2}+\frac{b+c}{Z^2}\right]\frac{a+c}{a+b}$$
$$=-\frac{a+b}{X^2}+\frac{(a+c)^2}{(a+b)Y^2}-\frac{(b-c)(b+c)}{(a+b)Z^2}.$$

We will show that $E'(b)\le 0$, that is

$$\frac{(b-c)(b+c)}{Z^2}\ge\frac{(a+c)^2}{Y^2}-\frac{(a+b)^2}{X^2},\qquad \frac{(b-c)(b+c)}{Z^2}\ge\frac{AB}{X^2Y^2},$$

where

$$A=(a+c)X-(a+b)Y,B=(a+c)X+(a+b)Y.$$

Since

$$A=(a+b)(a+c)(b-c)-k(b-c)=(b-c)(a^2+3-k),$$
$$B=(a+b)(a+c)(2a+b+c)+k(2a+b+c)=(2a+b+c)(a^2+3+k),$$

and

$$XY\ge[(a+b)(a+c)+2]^2=(a^2+3+k)^2,$$

it suffices to show that

$$\frac{b+c}{Z^2}\ge\frac{(2a+b+c)(a^2+3-k)}{(a^2+3+k)^3},$$

which is equivalent to

$$f(x)=\frac{a^2+3-k}{(a^2+3+k)^3},$$

where

$$f(x)=\frac{x}{(x^2+k)^2(2a+x)},\qquad x=b+c.$$

From

$$12-3(b+c)^2=4(ab+bc+ca)-3(b+c)^2\geq 4(b^2+2bc)-3(b+c)^2$$
$$=b^2+2bc-3c^2\geq 0,$$

we get $b+c\leq 2$, hence $x\in[a,2]$ with $a\geq 1$. We have

$$f'(x)=\frac{-A}{(x^2+k)^3(2a+x)^2},\ \ A=4x^3+6ax^2-2ka.$$

Since $A\geq 4a^3+6a^3-2ka=2a(5a^2-k)>0$, we have $f'(x)<0, f(x)$ is decreasing, hence

$$f(x)\geq f(2)=\frac{1}{(4+k)^2(a+1)}.$$

From $(2a-3)^2\geq 0$, we get

$$12a\leq 4a^2+9,\qquad a+1\leq\frac{4a^2+21}{12},\frac{1}{a+1}\geq\frac{12}{4a^2+21},$$

hence

$$f(x)\geq\frac{12}{(4+k)^2(4a^2+21)}.$$

So, it suffices to show that

$$\frac{12}{(4+k)^2(4a^2+21)}\geq\frac{a^2+3-k}{(a^2+3+k)^3}.$$

Denoting $t=a^2\geq 1$, the inequality is equivalent to $g(t)\geq 0$, where

$$g(t)=12(t+3+k)^3-(4+k)^2(4t+21)(t+3-k).$$

We have

$$g'(t)=36(t+3+k)^2-(4+k)^2(8t+33-4k),\ \ g''(t)$$
$$=72(t+3+k)-8(4+k)^2.$$

Since

$$g''(t)\geq 72(4+k)-8(4+k)^2=8(4+k)(5-k)>0,$$

$g'(t)$ is increasing, hence

$$g'(t)\geq g'(1)=36(4+k)^2-(4+k)^2(41-4k)=(4+k)^2(4k-5)>0,$$

therefore $g(t)$ is increasing and

$$g(t)\geq g(1)=12(4+k)^3-25(4+k)^2(4-k)=(4+k)^2(37k-52)>0.$$

Since $E'(b)\leq 0$, then $E(b)$ is decreasing and has the maximum value when b is minimum, hence when $b=\max\{c,a-c\}$. So, it suffices to consider the following two cases: $b=c$ and $b=a-c$.

Case 1: $b=c\geq\frac{a}{2}$. For $b=c:=z\geq\frac{a}{2}$, the equality constraint becomes $2az+z^2=3$, and from $3=2az+z^2\leq 4z^2+z^2=5z^2$, we get $5z^2-3\geq 0$. The inequality becomes

$$\frac{2}{(a+z)^2+k}+\frac{1}{4z^2+k}\leq\frac{3}{4+k},$$

which is equivalent to

$$\frac{32z^2}{5z^4+62z^2+45}+\frac{1}{5z^2+2}\le\frac{3}{6},$$
$$5z^6-13z^4+11z^2-3\ge 0,\qquad (z^2-1)^2(5z^2-3)\ge 0.$$

The last inequality is clearly true.

Case 2: $a=b+c$. We need to show that $(b+c)^2+bc=3$ implies

$$\frac{1}{(b+c)^2+k}+\frac{1}{(2b+c)^2+k}+\frac{1}{(b+2c)^2+k}\le\frac{3}{4+k}.$$

Let $x=b+c$. We have

$$bc=3-x^2,$$

and from $x^2\ge 4bc=4(3-x^2)$, we get

$$x^2\ge\frac{12}{5}.$$

Since

$$\frac{1}{(2b+c)^2+k}+\frac{1}{(b+2c)^2+k}=\frac{5(b+c)^2-2bc+2k}{[(2b+c)^2+k][(b+2c)^2+k]}\le$$
$$\le\frac{5(b+c)^2-2bc+2k}{[(2b+c)(b+2c)+k]^2}$$
$$=\frac{5x^2-2bc+2k}{(2x^2+bc+k)^2}=\frac{7x^2-6+2k}{(x^2+3+k)^2},$$

it suffices to show that

$$\frac{1}{x^2+k}+\frac{7x^2-6+2k}{(x^2+3+k)^2}\le\frac{3}{4+k},$$

i.e.

$$\frac{1}{5x^2+8}+\frac{35x^2-14}{(5x+23)^2}\le\frac{3}{28}.$$

Substituting

$$5x^2=12t,\qquad t\ge 1,$$

the inequality becomes as follows:

$$\frac{1}{4(3t+2)}+\frac{14(6t-1)}{(12t+23)^2}\le\frac{3}{28},$$
$$432t^3-744t^2+227t+85\ge 0,\qquad (t-1)(432t^2-312t-85)\ge 0.$$

Since $t\ge 1$, the last inequality is clearly true.

The proof is completed. For $k=\frac{8}{5}$, the equality occurs when $a=b=c=1$, and also when $\frac{a}{2}=b=c=\sqrt{\frac{3}{5}}$ (or any cyclic permutation).

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

For $a=b=x\in(0,\sqrt{3})$ and $c=\frac{3-x^2}{2x}$, the constraints are satisfied and the inequality becomes

$$\frac{1}{4x^2+k}+\frac{2}{\left(x+\frac{3-x^2}{2x}\right)^2+k}\le\frac{3}{4+k}.$$

Since $a+b>c$, we have $2x>\frac{3-x^2}{2x}$, then $x>\sqrt{\frac{3}{5}}$, and for $x \to \sqrt{\frac{3}{5}}$, we get the necessary condition $\frac{1}{\frac{12}{5}+k}+\frac{2}{\frac{27}{5}+k}\le\frac{3}{4+k}$, that is $k\ \ge\ \frac{8}{5}$.

To show that $\frac{8}{5}$ is the smallest positive value of k, we will prove the given inequality for $k=\frac{8}{5}$.

The given inequality is equivalent to:

$$\frac{(a+b)^2}{(a+b)^2+k}+\frac{(b+c)^2}{(b+c)^2+k}+\frac{(c+a)^2}{(c+a)^2+k}\ge\frac{12}{4+k}.$$

Let $2p=a+b+c\ge\sqrt{3(ab+bc+ca)}=3$ and $r=abc$.

By Cauchy – Schwarz's inequality, we have:

$$\sum_{cyc}\frac{(b+c)^2}{(b+c)^2+k}\ge\frac{\left[\sum_{cyc}(b+c)(5s+2a)\right]^2}{\sum_{cyc}(5s+2a)^2((b+c)^2+k)}\overset{?}{\ge}\frac{12}{4+k}$$

$$\Leftrightarrow 252+333p^2-100p^4-330pr\ge 0.$$

Let $a=y+z, b=z+x, c=x+y$. Since $ab+bc+ca=3$, we have $xy+yz+zx=3-p^2$,

and $r=abc=(x+y+z)(xy+yz+zx)-xyz=p(3-p^2)-xyz$, then the last inequality is equivalent to

$$252-657p^2+230p^4+330pxyz\ge 0 \qquad (*)$$

If $p^2\ge\frac{12}{5}$, we have $LHS_{(*)}\ge 252-657p^2+230p^4=(5p^2-12)(46p^2-21)\ge 0$.

If $\frac{9}{4}\le p^2\le\frac{12}{5}$. By the fourth degree Schur's inequality, we have

$$6pxyz\ge\left(p^2-\sum_{cyc}yz\right)\left(4\sum_{cyc}yz-p^2\right)=(2p^2-3)(12-5p^2)$$

Then:

$LHS_{(*)}\ge 252-657p^2+230p^4+55(2p^2-3)(12-5p^2)=16(12-5p^2)(4p^2-9)\ge 0,$

So, the proof is completed. The equality occurs when $a=b=c=1$.