UP.614 If $0 < a \leq b$ then:

$$\int_a^b \int_a^b \frac{dxdy}{x^2 - xy + y^2} + 3\left(\ln\left(\frac{b}{a}\right)\right)^2 \geq 16 \int_a^b \int_a^b \frac{dxdy}{(x+y)^2}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$\left(\ln\left(\frac{b}{a}\right)\right) = \ln\frac{b}{a} \cdot \ln\frac{b}{a} = \int_a^b \frac{1}{x} dx \cdot \int_a^b \frac{1}{y} dy = \int_a^b \int_a^b \frac{dxdy}{xy}$$

The inequality to prove can be written:

$$\int_a^b \int_a^b \frac{dxdy}{x^2 - xy + y^2} + 3\int_a^b \int_a^b \frac{dxdy}{xy} \geq \int_a^b \int_a^b \frac{16dxdy}{(x+y)^2}$$

$$\int_a^b \int_a^b \left(\frac{1}{x^2 - xy + y^2} + \frac{3}{xy}\right) dxdy \geq \int_a^b \int_a^b \frac{16}{(x+y)^2} dx\, dy$$

It is enough to prove that:

$$\frac{1}{x^2 - xy + y^2} + \frac{3}{xy} \geq \frac{16}{(x+y)^2}$$

$$\frac{xy + 3x^2 - 3xy + 3y^2}{xy(x^2 - xy + y^2)} \geq \frac{16}{x^2 + 2xy + y^2}$$

$$(3x^2 - 2xy + 3y^2)(x^2 + 2xy + y^2) \geq 16xy(x^2 - xy + y^2)$$

$$3x^4 + 6x^3y + 3x^2y^2 - 2x^3y - 4x^2y^2 - 2xy^3 + 3x^2y^2 +$$

$$+6xy^3 + 3y^4 \geq 16x^3y - 16x^2y^2 + 16xy^3$$

$$3x^4 + 3y^4 - 12x^3y - 12xy^3 + 18x^2y^2 \geq 0$$

$$x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4 \geq 0$$

$(x-y)^4 \geq 0$. **True!**

Equality holds for $a = b$.

Solution 2 by Tapas Das-India

$$\textit{We will show}:$$

$$\frac{1}{x^2 - xy + y^2} + \frac{3}{xy} \geq \frac{16}{(x+y)^2}$$

Proof: *We need to show*:

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$$\frac{1}{x^2-xy+y^2}+\frac{3}{xy}\geq\frac{16}{(x+y)^2}$$

$$\frac{1}{u^2-u+1}+\frac{3}{u}\overset{u=\frac{x}{y}>0}{\geq}\frac{16}{(u+1)^2}$$

$$(u+1)^2(3u^2-2u+3)\geq 16u(u^2-u+1)$$

$$3u^4+4u^3+2u^2+4u+3\geq 16u^3-16u^2+16u$$

$$3(u^4-4u^3+6u^2-4u+1)\geq 0$$

$$3(u-1)^4\geq 0\ true, so\ \frac{1}{x^2-xy+y^2}+\frac{3}{xy}\geq\frac{16}{(x+y)^2}$$

$$now\ \int_a^b\int_a^b\left(\frac{1}{x^2-xy+y^2}+\frac{3}{xy}\right)dxdy\geq 16\int_a^b\int_a^b\frac{dxdy}{(x+y)^2}$$

$$\int_a^b\int_a^b\frac{dxdy}{x^2-xy+y^2}+3\int_a^b\int_a^b\frac{dxdy}{xy}\geq 16\int_a^b\int_a^b\frac{dxdy}{(x+y)^2}$$

$$\int_a^b\int_a^b\frac{dxdy}{x^2-xy+y^2}+3\left(\ln\left(\frac{b}{a}\right)\right)^2\geq 16\int_a^b\int_a^b\frac{dxdy}{(x+y)^2}$$

Equality holds for a=b.