ROMANIAN MATHEMATICAL MAGAZINE

UP.613 Let be $A(1,2,0); B(2,0,1); C(3,3,3)$. Find the area of ΔABC.

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$\overrightarrow{AB} = (x_B - x_A)\vec{i} + (y_B - y_A)\vec{j} + (z_B - z_A)\vec{k}$$

$$\overrightarrow{AB} = (2-1)\vec{i} + (0-2)\vec{j} + (1-0)\vec{k}$$

$$\overrightarrow{AB} = \vec{i} - 2\vec{j} + \vec{k}$$

$$\overrightarrow{AC} = (x_C - x_A)\vec{i} + (y_C - y_A)\vec{j} + (z_C - z_A)\vec{k}$$

$$\overrightarrow{AC} = (3-1)\vec{i} + (3-2)\vec{j} + (3-0)\vec{k}$$

$$\overrightarrow{AC} = 2\vec{i} + \vec{j} + 3\vec{k}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 1 \\ 2 & 1 & 3 \end{vmatrix} = -6\vec{i} + 2\vec{j} + \vec{k} + 4\vec{k} - 3\vec{j} - \vec{i}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = -7\vec{i} - \vec{j} + 5\vec{k}$$

$$\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \sqrt{(-7)^2 + (-1)^2 + 5^2} = \sqrt{75} = 5\sqrt{3}$$

$$\text{Area}\,(\Delta ABC) = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{5\sqrt{3}}{2}$$

Solution 2 by José Luis Díaz – Barrero – Spain

The area of a triangle formed by two vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is given by the formula:

$$Area = \frac{1}{2}\left|\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|\right|$$

We have,

$$\overrightarrow{AB} = B - A = (2-1, 0-2, 1-0) = (1, -2, 1),$$

$$\overrightarrow{AC} = C - A = (3-1, 3-2, 3-0) = (2, 1, 3),$$

and

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 1 & -2 & 1 \\ 2 & 1 & 3 \end{vmatrix}$$

Expanding along the first row, we get

$$\overrightarrow{AB} \times \overrightarrow{AC} = i\begin{vmatrix} -2 & 1 \\ 1 & 3 \end{vmatrix} - j\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} + k\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix}$$

$$= i\big((-2)(3) - (1)(1)\big) - j\big((1)(3) - (1)(2)\big) + k\big((1)(1) - (-2)(2)\big)$$

$$= i(-6-1) - j(3-2) + k(1+4)$$

$$= -7\boldsymbol{i} - 1\boldsymbol{j} + 5\boldsymbol{k} = (-7, -1, 5)$$

Now, we find the length or norm of the cross product vector:

$$\left|\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|\right| = \sqrt{(-7)^2 + (-1)^2 + (5)^2}$$

$$= \sqrt{75} = 5\sqrt{3}$$

Plugging the magnitude back into the triangle formula, we obtain

$$Area = \frac{1}{2} \cdot 5\sqrt{3} = \frac{5\sqrt{3}}{2}$$

Solution 3 by Marin Chirciu-Romania

We use: $S(\Delta ABC) = \frac{1}{2}\left|\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|\right|$.

In our case:

$$\overrightarrow{AB} = (2-1)\vec{i} + (0-2)\vec{j} + (1-0)\vec{k} = \vec{i} - 2\vec{j} + \vec{k} \Rightarrow$$

$$\Rightarrow \left|\left|\overrightarrow{AB}\right|\right| = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{6}.$$

$$\overrightarrow{AC} = (3-1)\vec{i} + (3-2)\vec{j} + (3-0)\vec{k} = 2\vec{i} + \vec{j} + 3\vec{k} \Rightarrow$$

$$\Rightarrow \left|\left|\overrightarrow{AC}\right|\right| = \sqrt{2^2 + 1^2 + 3^2} = \sqrt{14}$$

$\left|\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|\right| = \left|\left|\overrightarrow{AB}\right|\right| \cdot \left|\left|\overrightarrow{AC}\right|\right| \cdot \sin\angle\left(\overrightarrow{AB}, \overrightarrow{AC}\right) = \sqrt{6} \cdot \sqrt{14} \cdot \frac{5}{2\sqrt{7}} = 5\sqrt{3}$, **see:**

$$\cos\angle\left(\overrightarrow{AB}, \overrightarrow{AC}\right) = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\left|\left|\overrightarrow{AB}\right|\right| \cdot \left|\left|\overrightarrow{AC}\right|\right|} = \frac{\left(\vec{i} - 2\vec{j} + \vec{k}\right) \cdot \left(2\vec{i} + \vec{j} + 3\vec{k}\right)}{\sqrt{6} \cdot \sqrt{14}} = \frac{1 \cdot 2 + (-2) \cdot 1 + 1 \cdot 3}{2 \cdot \sqrt{21}}$$

$$= \frac{3}{2\sqrt{21}} \Rightarrow \sin\angle\left(\overrightarrow{AB}, \overrightarrow{AC}\right) = \sqrt{1 - \left(\frac{3}{2\sqrt{21}}\right)^2} = \frac{5}{2\sqrt{7}}$$

We deduce $S(\Delta ABC) = \frac{1}{2} \cdot 5\sqrt{3} = \frac{5\sqrt{3}}{2}$