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**UP.613.** *Proposed by Daniel Sitaru, Romania.* Let $A(1,2,0); B(2,0,1); C(3,3,3)$ be points in $\mathbb{RT}^3$. Find the area of $\triangle ABC$.

*Solution by José Luis Díaz-Barrero, Barcelona, Spain.* The area of a triangle formed by two vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is given by the formula:

$$\text{Area} = \frac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\|$$

We have,

$$\begin{aligned}\overrightarrow{AB} &= B - A = (2-1, 0-2, 1-0) = (1,-2,1),\\ \overrightarrow{AC} &= C - A = (3-1, 3-2, 3-0) = (2,1,3),\end{aligned}$$

and

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ 2 & 1 & 3 \end{vmatrix}$$

Expanding along the first row, we get

$$\begin{aligned}\overrightarrow{AB} \times \overrightarrow{AC} &= \mathbf{i}\begin{vmatrix} -2 & 1 \\ 1 & 3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix}\\ &= \mathbf{i}((-2)(3) - (1)(1)) - \mathbf{j}((1)(3) - (1)(2)) + \mathbf{k}((1)(1) - (-2)(2))\\ &= \mathbf{i}(-6-1) - \mathbf{j}(3-2) + \mathbf{k}(1+4)\\ &= -7\mathbf{i} - 1\mathbf{j} + 5\mathbf{k} = (-7,-1,5)\end{aligned}$$

Now, we find the length or norm of the cross product vector:

$$\begin{aligned}\|\overrightarrow{AB} \times \overrightarrow{AC}\| &= \sqrt{(-7)^2 + (-1)^2 + (5)^2}\\ &= \sqrt{75} = 5\sqrt{3}\end{aligned}$$

Plugging the magnitude back into the triangle formula, we obtain

$$\text{Area} = \frac{1}{2} \cdot 5\sqrt{3} = \frac{5\sqrt{3}}{2}$$