ROMANIAN MATHEMATICAL MAGAZINE

UP.612 If $f:\mathbb{R}_+^* \to \mathbb{R}_+^*$ is a continuous function and $(a_n)_{n\ge1}$ is defined by $a_1 = a_2 = 1$,

$$a_{n+1} = \sum_{k=1}^{n}\frac{a_k}{k}, \forall n \ge 2 \text{ and } (x_n)_{n\ge1}, x_n = \sum_{k=1}^{n}\frac{1}{a_k}, \text{ find}$$

$$\lim_{n\to\infty}\frac{1}{n}\int_{e^{x_n}}^{e^{x_{n+1}}}\frac{f(x-e^{x_n})}{f(e^{x_{n+1}}-x)+f(x-e^{x_n})}dx$$

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Solution 1 by proposer

If $a,b\in\mathbb{R}_+^* a<b$, then $I=\int_a^b\frac{f(x-a)}{f(b-x)+f(x-a)}dx=\frac{b-a}{2}$.

$$\text{So, } I_n=\int_{e^{x_n}}^{e^{x_{n+1}}}\frac{f(x-e^{x_n})}{f(e^{x_{n+1}}-x)+f(x-e^{x_n})}dx=\frac{e^{x_{n+1}}-e^{x_n}}{2}=$$

$$=\frac{1}{2}\cdot e^{x_n}\cdot\left(e^{x_{n+1}-x_n}-1\right)=\frac{1}{2}\cdot e^{x_n}\cdot\left(e^{\frac{1}{a_{n+1}}}-1\right)=\frac{1}{2}\cdot\frac{e^{x_n}}{a_{n+1}}\cdot\frac{e^{\frac{1}{a_{n+1}}}-1}{\frac{1}{a_{n+1}}},\ (2).$$

By induction we obtain $a_n=\frac{n}{2}, \forall n\ge2$, so $x_n=2\cdot\sum_{k=1}^{n}\frac{1}{k}-1$, then by (2)

$$\frac{1}{n}I_n=\frac{1}{2}\cdot\frac{e^{x_n}}{n^2}\cdot\left(e^{\frac{1}{a_n}}-1\right)\cdot n=\frac{1}{2}\cdot\frac{e^{x_n}}{n^2}\cdot\left(e^{\frac{2}{n+1}}-1\right)\cdot n=$$

$$=\frac{1}{2}e^{x_n-2\ln n}\cdot\left(e^{\frac{1}{n+1}}+1\right)\left(e^{\frac{1}{n+1}}-1\right)\cdot n=$$

$$=\frac{1}{2}\cdot\left(e^{\frac{1}{n+1}}+1\right)\cdot e^{2\sum_{k=1}^{n}\frac{1}{k}-2\ln n-1}\cdot\frac{e^{\frac{1}{n+1}}}{\frac{1}{n+1}}\cdot\frac{n}{n+1}=\frac{1}{2}\left(e^{\frac{1}{n+1}}+1\right)\cdot e^{2\gamma_n-1}\cdot\frac{e^{\frac{1}{n+1}}-1}{\frac{1}{n+1}}\cdot\frac{n}{n+1}\quad(3)$$

Since $\gamma_n=-\ln n+\sum_{k=1}^{n}\frac{1}{k}$ and $\lim_{n\to\infty}\gamma_n=\gamma$ – Euler- Mascheroni constant and

$$\lim_{n\to\infty}\frac{e^{\frac{1}{n+1}}-1}{\frac{1}{n+1}}=1, \text{ hence } \lim_{n\to\infty}\frac{1}{n}I_n=\frac{1}{2}\cdot(1+1)\cdot e^{2\gamma-1}\cdot1\cdot1=e^{2\gamma-1}.$$

Solution 2 by Omkumar Rao-India

$$I_n=\int_{e^{x_n}}^{e^{x_{n+1}}}\frac{f(x-e^{x_n})}{f(e^{x_{n+1}}-x)+f(x-e^{x_n})}dx\Rightarrow(1)$$

$$(x\to e^{x_n}+e^{x_{n+1}}-x):$$

$$I_n=\int_{e^{x_n}}^{e^{x_{n+1}}}\frac{f(e^{x_{n+1}}-x)}{f(x-e^{x_n})+f(e^{x_{n+1}}-x)}dx\Rightarrow(2)$$

$$2I_n = \int_{e^{x_n}}^{e^{x_{n+1}}} 1\,dx, \qquad I_n = \frac{(e^{x_{n+1}} - e^{x_n})}{2}, \qquad L = \lim_{n\to\infty} \frac{1}{2n}(e^{x_{n+1}} - e^{x_n})$$

$$a_{n+1} = \sum_{k=1}^{n} \frac{a_k}{k} = \left(\sum_{k=1}^{n-1} \frac{a_k}{k}\right) + \frac{a_n}{n} = a_n + \frac{a_n}{n}$$

$$a_n = \frac{n}{2}$$

$$x_n = \sum_{k=1}^{n} \frac{1}{a_k} = \frac{1}{a_1} + \sum_{k=2}^{n} \frac{2}{k}, \qquad x_n = 1 + 2(H_n - 1) = 2H_n - 1$$

$$H_n = \gamma + \ln n + \frac{1}{2n} + O\left(\frac{1}{n^2}\right)$$

$$x_n = 2\gamma + 2\ln n + \frac{1}{n} + 2O\left(\frac{1}{n^2}\right) - 1$$

$$e^{x_n} = e^{(2\gamma-1)+2O\left(\frac{1}{n^2}\right)+\ln n^2} = n^2 e^{2\gamma-1} e^{2O\left(\frac{1}{n^2}\right)}$$

$$e^{2O\left(\frac{1}{n^2}\right)} = 1 + 2O\left(\frac{1}{n^2}\right) \approx 1, \quad e^{x_n} = n^2 e^{2\gamma-1}$$

$$x_{n+1} = x_n + \frac{1}{a_{n+1}}$$

$$L = \lim_{n\to\infty} \frac{e^{x_n}\left(e^{\frac{2}{n+1}} - 1\right)}{2n}, \qquad L = \lim_{n\to\infty} \frac{n^2 e^{2\gamma-1}(2)}{2n(n+1)}$$

$$L = e^{2\gamma-1}$$