UP.611 If $f:\mathbb{R}_+^*\to\mathbb{R}_+^*$ is a continuous function and $(x_n)_{n\ge1}, x_n=\sum_{k=1}^{n}\frac{1}{k}$, find

$$\lim_{n\to\infty}\int_{e^{x_n}}^{e^{x_{n+1}}}\frac{f(x-e^{x_n})}{f(e^{x_{n+1}}-x)+f(x-e^{x_n})}dx$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution 1 by proposers

If $a,b\in\mathbb{R}_+^*, a<b$, then $I=\int_a^b\frac{f(x-a)}{f(b-x)+f(x-a)}dx=\frac{b-a}{2}$.

So, $\int_{e^{x_n}}^{e^{x_{n+1}}}\frac{f(x-e^{x_n})}{f(e^{x_{n+1}}-x)+f(x-e^{x_n})}dx=\frac{e^{x_{n+1}}-e^{x_n}}{2}=$

$$=\frac{1}{2}\cdot e^{x_n}\cdot\left(e^{x_{n+1}-x_n}-1\right)=\frac{1}{2}\cdot e^{x_n}\cdot(x_{n+1}-x_n)\cdot\frac{e^{x_{n+1}-x_n}-1}{x_{n+1}-x_n}=$$

$$=\frac{1}{2}\cdot e^{x_n}\cdot\frac{1}{n+1}\cdot\frac{e^{\frac{1}{n+1}}-1}{\frac{1}{n+1}}=$$

$$=\frac{1}{2}\cdot\frac{e^{x_n}}{n}\cdot\frac{n}{n+1}\cdot\frac{e^{\frac{1}{n+1}}-1}{\frac{1}{n+1}}=\frac{1}{2}\cdot e^{-\ln n+x_n}\cdot\frac{n}{n+1}\cdot\frac{e^{\frac{1}{n+1}}-1}{\frac{1}{n+1}}$$

$$=\frac{1}{2}\cdot e^{\gamma_n}\cdot\frac{n}{n+1}\cdot\frac{e^{\frac{1}{n+1}}-1}{\frac{1}{n+1}},$$

Therefore:

$$\lim_{n\to\infty}I_n=\frac{1}{2}\cdot e^{\gamma}\cdot1\cdot1=\frac{1}{2}e^{\gamma}$$

$\gamma_n=-\ln n+\sum_{k=1}^{n}\frac{1}{k}$ and $\lim_{n\to\infty}\gamma_n=\gamma$ Euler – Mascheroni constant.

Solution 2 by Omkumar Rao-India

$$I_n=\int_{e^{x_n}}^{e^{x_{n+1}}}\frac{f(x-e^{x_n})}{f(e^{x_{n+1}}-x)+f(x-e^{x_n})}dx\qquad\to(1)$$

$$(x\to e^{x_n}+e^{x_{n+1}}-x)$$

$$I_n=\int_{e^{x_n}}^{e^{x_{n+1}}}\frac{f(e^{x_{n+1}}-x)}{f(x-e^{x_n})+f(e^{x_{n+1}}-x)}dx\to(2)$$

$$2I_n = \int_{e^{x_n}}^{e^{x_{n+1}}} 1\,dx\,, \qquad I_n = \frac{e^{x_{n+1}} - e^{x_n}}{2}$$

$$e^{x_{n+1}} - e^{x_n} = e^{x_n}\left(e^{\frac{1}{n+1}} - 1\right) = e^{x_n}\left(\frac{1}{n+1} + O\left(\frac{1}{n^2}\right)\right)$$

$$x_n = \ln n + \gamma + \frac{1}{2n} + O\left(\frac{1}{n^2}\right)$$

where γ is Euler Mascheroni constant.

$$I_n = ne^{\gamma}\left(1 + O\left(\frac{1}{n}\right)\right)\left(\frac{1}{n+1}\right)$$

$$I_n = \frac{e^{\gamma}}{2}.\frac{n}{n+1}.\left(1 + O\left(\frac{1}{n}\right)\right)$$

$$As\; n \to \infty\,, \frac{n}{n+1} \to 1\,, 1 + O\left(\frac{1}{n}\right) \to 1$$

$$\lim_{n\to\infty} \int_{e^{x_n}}^{e^{x_{n+1}}} \frac{f(x - e^{x_n})}{f(e^{x_{n+1}} - x) + f(x - e^{x_n})}\,dx = \frac{e^{\gamma}}{2}$$