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UP.610 If $f:\mathbb{R}_+^*\to\mathbb{R}_+^*$ is a continuous function and $\gamma_n=-\ln n+\sum_{k=1}^{n}\frac{1}{k}$, find

$$\lim_{n\to\infty} n\int_{\gamma}^{\gamma_n}\frac{f(x-\gamma)}{f(\gamma_n-x)+f(x-\gamma)}dx$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution 1 by proposers

Let $a,b\in\mathbb{R}_+^*, a<b, I=\int_a^b\frac{f(x-a)}{f(b-x)+f(x-a)}dx=\frac{b-a}{2}$

$\lim_{n\to\infty}\gamma_n=\gamma$ – Euler Mascheroni constant

Therefore, $I_n=\int_{\gamma}^{\gamma_n}\frac{f(x-\gamma)}{f(\gamma_n-x)+f(x-\gamma)}dx=\frac{\gamma_n-\gamma}{2}$, then

$$\lim_{n\to\infty} nI_n=\frac{1}{2}\lim_{n\to\infty}\frac{\gamma_n-\gamma}{\frac{1}{n}}\overset{Cesaro-Stolz}{\underset{\left(\frac{0}{0}\right)}{=}}\frac{1}{2}\lim_{n\to\infty}\frac{\gamma_{n+1}-\gamma_n}{\frac{1}{n+1}-\frac{1}{n}}=\frac{1}{2}\lim_{n\to\infty}\frac{\gamma_n-\gamma_{n+1}}{\frac{1}{n(n+1)}}=$$

$$=\frac{1}{2}\lim_{n\to\infty}\frac{-\frac{1}{n+1}+\ln\frac{n+1}{n}}{\frac{1}{n(n+1)}}=\frac{1}{2}\lim_{n\to\infty}\frac{-1+(n+1)\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}}\overset{\left(x=\frac{1}{n}\right)}{=}$$

$$=\frac{1}{2}\lim_{\substack{x\to0\\x>0}}\frac{-x+(1+x)\ln(1+x)}{x^2}\overset{L'H}{\underset{\left(\frac{0}{0}\right)}{=}}$$

$$=\frac{1}{2}\lim_{\substack{x\to0\\x>0}}\frac{-1+\ln(1+x)+(1+x)\cdot\frac{1}{1+x}}{2x}=\frac{1}{4}\lim_{\substack{x\to0\\x>0}}\ln(1+x)^{\frac{1}{x}}=\frac{1}{4}\ln e=\frac{1}{4}$$

Solution 2 by Omkumar Rao-India

$$I_n=\int_{\gamma}^{\gamma_n}\frac{f(x-\gamma)}{f(\gamma_n-x)+f(x-\gamma)}dx\implies(1)$$

$$(x\longrightarrow\gamma+\gamma_n-x):\quad I_n=\int_{\gamma}^{\gamma_n}\frac{f(\gamma_n-x)}{f(x-\gamma)+f(\gamma_n-x)}dx\implies(2)$$

$$2I_n=\int_{\gamma}^{\gamma_n}1\,dx$$

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$$I_n=\frac{(\gamma_n-\gamma)}{2},\qquad L=\lim_{n\to\infty} n\frac{(\gamma_n-\gamma)}{2}$$

$$\gamma_n=-\ln n+\sum_{k=1}^{n}\frac{1}{k},\qquad \gamma_n-\gamma=-\ln n+\sum_{k=1}^{n}\frac{1}{k}-\gamma$$

$$\sum_{k=1}^{n}\frac{1}{k}:H_n\ as\ n\to\infty$$

$$H_n=\ln n+\gamma+\frac{1}{2n}+O\left(\frac{1}{n^2}\right)$$

$$\lim_{n\to\infty} n\frac{-\gamma-\ln n+\ln n+\gamma+\frac{1}{2n}+O\left(\frac{1}{n^2}\right)}{2}=\lim_{n\to\infty} n\frac{\frac{1}{2n}+O\left(\frac{1}{n^2}\right)}{2}$$

$$L=\frac{1}{4}$$