ROMANIAN MATHEMATICAL MAGAZINE

UP.609 If $f:[a,b]\to(0,\infty)$ is continuous functions such that

$$f(a+b-x)+f(x)=c,\forall x\in[a,b] \text{ and } a+b=\frac{\pi}{2}, \text{ find:}$$

$$\int_a^b \frac{\sin^n x+f(x)+d}{\sin^n x+\cos^n x+c+2d}dx,$$

where $n\in\mathbb{N}^*$ and $d\geq 0$.

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution by proposers

$$x=u(t)=\frac{\pi}{2}-t, u'(t)=-1, u(0)=\frac{\pi}{2}, u\left(\frac{\pi}{2}\right)=0, u(a)=b, u(b)=a.$$

Hence, $I=\int_a^b \frac{\sin^n x+f(x)+d}{\sin^n x+\cos^n x+c+2d}dx=\int_b^a \frac{\sin^n\left(\frac{\pi}{2}-t\right)+f\left(\frac{\pi}{2}-t\right)+d}{\sin^n\left(\frac{\pi}{2}-t\right)+\cos^n\left(\frac{\pi}{2}-t\right)+c+2d}(-1)dt=$

$$=\int_a^b \frac{\cos^n t+\left(c-f(t)\right)+d}{\cos^n t+\sin^n t+c+2d}dt=\int_a^b \frac{\cos^n x+c-f(x)+d}{\cos^n x+\sin^n x+c+2d}dx;$$

$2I=\int_a^b \frac{\sin^n x+f(x)+d+\cos^n x+c-f(x)+d}{\cos^n x+\sin^n x+c+2d}dx=\int_a^b dx=b-a$; therefore $I=\frac{b-a}{2}$