ROMANIAN MATHEMATICAL MAGAZINE

UP.608 If $f:\mathbb{R}\to(1,\infty)$ and $g:\mathbb{R}\to\mathbb{R}$ are continuous functions, and $(L_n)_{n\ge0}$ is Lucas sequence, and $y_n=\sqrt[n]{(2n-1)!!\,L_n}, n\in\mathbb{N}^*-\{1\}$, find:

$$\lim_{n\to\infty}\int_{y_n}^{y_{n+1}}\frac{\left(f(x-y_n)\right)^{g(y_{n+1}-x)}}{\left(f(y_{n+1}-x)\right)^{g(x-y_n)}+\left(f(x-y_n)\right)^{g(y_{n+1}-x)}}dx.$$

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Solution by proposers

Lemma 1. If $f:\mathbb{R}\to(1,\infty), g:\mathbb{R}\to\mathbb{R}$ are continuous, and $(y_n)_{n\ge1}$ is an increasing sequence, and

$$I_n=\int_{y_n}^{y_{n+1}}\frac{\left(f(x-y_n)\right)^{g(y_{n+1}-x)}}{\left(f(y_{n+1}-x)\right)^{g(x-y_n)}+\left(f(x-y_n)\right)^{g(y_{n+1}-x)}}dx, \forall n\ge 2, \text{ then:}$$

$$\lim_{n\to\infty}I_n=\frac{1}{2}\lim_{n\to\infty}(y_{n+1}-y_n).$$

Proof.

If $a,b\in\mathbb{R}, a<b$ and $I=\int_a^b\frac{\left(f(x-a)\right)^{g(b-x)}}{\left(f(b-x)\right)^{g(x-a)}+\left(f(x-a)\right)^{g(b-x)}}dx$,

$x=u(t)=a+b-t, u'(t)=-1, u(a)=b, u(b)=a$, we obtain that

$$I=\int_b^a\frac{\left(f(b-x)\right)^{g(x-a)}}{\left(f(x-a)\right)^{g(b-x)}+\left(f(b-x)\right)^{g(x-a)}}dx,$$

$$2I=\int_a^b\frac{\left(f(b-x)\right)^{g(x-a)}+\left(f(x-a)\right)^{g(b-x)}}{\left(f(x-a)\right)^{g(b-x)}+\left(f(b-x)\right)^{g(x-a)}}dx=\int_a^b dx=b-a\Rightarrow I=\frac{1}{2}(b-a). \text{ So,}$$

$$I_n=\int_{y_n}^{y_{n+1}}\frac{\left(f(x-y_n)\right)^{g(y_{n+1}-x)}}{\left(f(y_{n+1}-x)\right)^{g(x-y_n)}+\left(f(x-y_n)\right)^{g(y_{n+1}-x)}}dx=\frac{1}{2}(y_{n+1}-y_n), \forall n\ge 2.$$

Hence: $\lim\limits_{n\to\infty}I_n=\frac{1}{2}\lim\limits_{n\to\infty}(y_{n+1}-y_n)$.

Lemma 3.

$\lim\limits_{n\to\infty}\left(\sqrt[n+1]{(2n+1)!!\,F_{n+1}}-\sqrt[n]{(2n-1)!!\,F_n}\right)=\frac{2\alpha}{e}$, where $\alpha=\frac{1+\sqrt{5}}{2}$, and

$(F_n)_{n\ge0}, F_0=0, F_1=1, F_{n+2}=F_{n+1}+F_n, \forall n\in\mathbb{N}$ is Fibonacci sequence,

Proof.

$$F_{n+2}-F_{n+1}-F_n=0, \forall n\in\mathbb{N}^* \Leftrightarrow \frac{F_{n+2}}{F_n}-\frac{F_{n+1}}{F_n}-1=0 \Leftrightarrow$$

$$\Leftrightarrow \frac{F_{n+2}}{F_{n+1}}\cdot\frac{F_{n+1}}{F_n}-\frac{F_{n+1}}{F_n}-1=0$$

$\Rightarrow \lim\limits_{n\to\infty}\left(\frac{F_{n+2}}{F_{n+1}}\cdot\frac{F_{n+1}}{F_n}-\frac{F_{n+1}}{F_n}-1\right)=0 \Leftrightarrow x^2-x-1=0$, where $x=\lim\limits_{n\to\infty}\frac{F_{n+1}}{F_n}$, so $x=\frac{1\pm\sqrt{5}}{2}$

Since $\frac{F_{n+1}}{F_n}>0, x=\lim\limits_{n\to\infty}\frac{F_{n+1}}{F_n}=\frac{\sqrt{5}+1}{2}=\alpha$.

$$\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!\,F_n}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{(2n-1)!!\,F_n}{n^n}}=\lim_{n\to\infty}\frac{(2n+1)!!\,F_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{(2n-1)!!\,F_n}=$$

$$=\lim_{n\to\infty}\frac{2n+1}{n+1}\cdot\frac{F_{n+1}}{F_n}\left(\frac{n}{n+1}\right)^n=2\cdot\alpha\cdot\frac{1}{e}=\frac{2\alpha}{e}.$$

(1) $\sqrt[n+1]{(2n+1)!!\,F_{n+1}}-\sqrt[n]{(2n-1)!!\,F_n}=\sqrt[n]{(2n-1)!!\,F_n}(u_n-1)=$

$$=\sqrt[n]{(2n-1)!!\,F_n}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n=\frac{\sqrt[n]{(2n-1)!!\,F_n}}{n}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n, \forall n\ge 2,$$

$$u_n=\frac{\sqrt[n+1]{(2n+1)!!\,F_{n+1}}}{\sqrt[n]{(2n-1)!!\,F_n}}$$

$\lim\limits_{n\to\infty}u_n=\lim\limits_{n\to\infty}\left(\frac{\sqrt[n+1]{(2n+1)!!F_{n+1}}}{n+1}\cdot\frac{n}{\sqrt[n]{(2n-1)!!F_n}}\cdot\frac{n+1}{n}\right)=\frac{2\alpha}{e}\cdot\frac{e}{2\alpha}\cdot 1=1$, so $\lim\limits_{n\to\infty}\frac{u_n-1}{\ln u_n}=1$,

$$\lim_{n\to\infty}u_n^n=\lim_{n\to\infty}\frac{(2n+1)!!\,F_{n+1}}{(2n-1)!!\,F_n}\cdot\frac{1}{\sqrt[n+1]{(2n+1)!!\,F_{n+1}}}=$$

$$=\lim_{n\to\infty}\frac{2n+1}{n+1}\cdot\frac{F_{n+1}}{F_n}\cdot\frac{n+1}{\sqrt[n+1]{(2n+1)!!\,F_{n+1}}}$$

$=2\cdot\alpha\cdot\frac{e}{2\alpha}=e$. Therefore:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{(2n+1)!!\,F_{n+1}}-\sqrt[n]{(2n-1)!!\,F_n}\right)=\frac{2\alpha}{e}\cdot 1\cdot\ln\left(\lim_{n\to\infty}u_n^n\right)=\frac{2\alpha}{e}\cdot\ln e=$$

$$=\frac{2\alpha}{e}\cdot 1=\frac{2\alpha}{e}.$$

Analogously, $\lim\limits_{n\to\infty}\left(\sqrt[n+1]{(2n+1)!!\,L_{n+1}}-\sqrt[n]{(2n-1)!!\,L_n}\right)=\frac{2\alpha}{e}$

Solution.

By lemma 1: $\lim_{n\to\infty} I_n = \frac{1}{2}\lim_{n\to\infty}(y_{n+1} - y_n) = \frac{1}{2}\cdot\frac{2\alpha}{e} = \frac{\alpha}{e}, \alpha = \frac{1+\sqrt{5}}{2}$ **where taking account by lemma 3:** $\lim_{n\to\infty}(y_{n+1} - y_n) = \lim_{n\to\infty}\left(\sqrt[n+1]{(2n+1)!\,L_{n+1}} - \sqrt[n]{(2n-1)!!\,L_n}\right) = \frac{2\alpha}{e}$.