ROMANIAN MATHEMATICAL MAGAZINE

UP.607 Calculate the integral:

$$\int_0^1 \frac{x\ln x}{(x+1)(x^2+1)}dx$$

Proposed by Vasile Mircea Popa – Romania

Solution 1 by proposer

Let us denote: $A=\int_0^1 \frac{x\ln x}{(x+1)(x^2+1)}dx$

We have, successively:

$$A=\int_0^1 \frac{(1-x)x\ln x}{1-x^4}dx=\int_0^1 \frac{x\ln x}{1-x^4}dx-\int_0^1 \frac{x^2\ln x}{1-x^4}dx$$

$$A=\int_0^1 \sum_{n=0}^{\infty} x^{4n+1}\ln x\,dx-\int_0^1 \sum_{n=0}^{\infty} x^{4n+2}\ln x\,dx=$$

$$=\sum_{n=0}^{\infty}\left(\int_0^1 x^{4n+1}\ln x\,dx-\int_0^1 x^{4n+2}\ln x\,dx\right)$$

We will use the following relationship:

$\int_0^1 x^a\ln x\,dx=-\frac{1}{(a+1)^2}$, where $a\in\mathbb{R}, a\geq 0$

We obtain:

$$A=\sum_{n=0}^{\infty}\left[\frac{1}{(4n+3)^2}-\frac{1}{(4n+2)^2}\right]=\sum_{n=0}^{\infty}\left[\frac{\frac{1}{16}}{\left(n+\frac{3}{4}\right)^2}-\frac{\frac{1}{16}}{\left(n+\frac{2}{4}\right)^2}\right]$$

We now use the following relationship

$$\Psi_1(x)=\sum_{n=0}^{\infty}\frac{1}{(x+n)^2}$$

where $\Psi_1(x)$ is the trigamma function.

We obtained the value of the integral A:

$$A=\frac{1}{16}\left[\Psi_1\left(\frac{3}{4}\right)-\Psi_1\left(\frac{1}{2}\right)\right]$$

The following special value is known:

$$\Psi_1\left(\frac{1}{2}\right)=\frac{\pi^2}{2}$$

Result:

$$A=\frac{1}{16}\left[\Psi_1\left(\frac{3}{4}\right)-\frac{\pi^2}{2}\right]$$

Thus, the problem is solved.

Solution 2 by Omkumar Rao-India

$$I=\frac{1}{2}\int_0^1\left(\frac{-\ln x}{x+1}+\frac{\ln x}{x^2+1}+\frac{x\ln x}{x^2+1}\right)dx$$

$$A=\int_0^1\frac{\ln x}{x+1}dx,\qquad B=\int_0^1\frac{\ln x}{x^2+1}dx$$

$$C=\int_0^1\frac{x\ln x}{x^2+1}dx$$

$$I=\frac{1}{2}(-A+B+C)$$

$$A=\sum_{n=0}^{\infty}(-1)^n\int_0^1 x^n\ln x\,dx$$

$$A=-\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2}=-\frac{1}{2}\zeta(2)=-\frac{\pi^2}{12}$$

$$B=\sum_{r=0}^{\infty}(-1)^r\int_0^1 x^{2r}\ln x\,dx=-\sum_{r=0}^{\infty}\frac{(-1)^r}{(2r+1)^2}=-G$$

$$where\ G\ denotes\ Catalan's\ constant.$$

$$C=\int_0^1\frac{x\ln x}{x^2+1}dx\Rightarrow(x\longrightarrow x^2)$$

$$C=\frac{1}{4}\int_0^1\frac{\ln x}{1+x}dx=\frac{1}{4}A=-\frac{\pi^2}{48}$$

$$I=\frac{1}{2}\left(\frac{\pi^2}{12}-G-\frac{\pi^2}{48}\right)$$

$$I=\frac{\pi^2}{32}-\frac{G}{2}$$