ROMANIAN MATHEMATICAL MAGAZINE

UP.606 If $f:\mathbb{R}\to(1,\infty)$ and $g:\mathbb{R}\to\mathbb{R}$ are continuous functions, and $y_n=\sqrt[n]{n!\,F_n}, n\in\mathbb{N}^*-\{1\}$, where $(F_n)_{n\ge0}$ is Fibonacci sequence, find:

$$\lim_{n\to\infty}\int_{y_n}^{y_{n+1}}\frac{\left(f(x-y_n)\right)^{g(y_{n+1}-x)}}{\left(f(y_{n+1}-x)\right)^{g(x-y_n)}+\left(f(x-y_n)\right)^{g(y_{n+1}-x)}}dx$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution by proposers

Lemma 1. If $f:\mathbb{R}\to(1,\infty)$, $g:\mathbb{R}\to\mathbb{R}$ are continuous, and $(y_n)_{n\ge1}$ is an increasing sequence, and

$$I_n=\int_{y_n}^{y_{n+1}}\frac{\left(f(x-y_n)\right)^{g(y_{n+1}-x)}}{\left(f(y_{n+1}-x)\right)^{g(x-y_n)}+\left(f(x-y_n)\right)^{g(y_{n+1}-x)}}dx,\forall n\ge2$$

then

$$\lim_{n\to\infty}I_n=\frac{1}{2}\lim_{n\to\infty}(y_{n+1}-y_n).$$

Proof.

If $a,b\in\mathbb{R}, a<b$ and $I=\int_a^b\frac{(f(x-a))^{g(b-x)}}{(f(b-x))^{g(x-a)}+(f(x-a))^{g(b-x)}}dx$,

$x=u(t)=a+b-t, u'(t)=-1, u(a)=b, u(b)=a$, we obtain that

$$I=\int_b^a\frac{\left(f(b-x)\right)^{g(x-a)}}{\left(f(x-a)\right)^{g(b-x)}+\left(f(b-x)\right)^{g(x-a)}}dx,$$

$2I=\int_a^b\frac{(f(b-x))^{g(x-a)}+(f(x-a))^{g(b-x)}}{(f(x-a))^{g(b-x)}+(f(b-x))^{g(x-a)}}dx=\int_a^b dx=b-a\Rightarrow I=\frac{1}{2}(b-a)$. So,

$$I_n=\int_{y_n}^{y_{n+1}}\frac{\left(f(x-y_n)\right)^{g(y_{n+1}-x)}}{\left(f(y_{n+1}-x)\right)^{g(x-y_n)}+\left(f(x-y_n)\right)^{g(y_{n+1}-x)}}dx=\frac{1}{2}(y_{n+1}-y_n),\forall n\ge2.$$

Hence: $\lim\limits_{n\to\infty}I_n=\frac{1}{2}\lim\limits_{n\to\infty}(y_{n+1}-y_n)$.

Lemma 2.

$$\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!\,L_{n+1}}-\sqrt[n]{n!\,L_n}\right)=\frac{\alpha}{e}, \text{ where } \alpha=\frac{1+\sqrt5}{2} \text{ and}$$

$(L_n)_{n\ge0}, L_0=2, L_1=1, L_{n+2}=L_{n+1}+L_n, \forall n\in\mathbb{N}$ is Lucas sequence.

Proof.

$$L_{n+2}-L_{n+1}-L_n=0,\forall n\in\mathbb{N}^*\Leftrightarrow\frac{L_{n+2}}{L_n}-\frac{L_{n+1}}{L_n}-1=0\Leftrightarrow\frac{L_{n+2}}{L_{n+1}}\cdot\frac{L_{n+1}}{L_n}-\frac{L_{n+1}}{L_n}-1=0$$

$$\Rightarrow$$

$$\Rightarrow\lim_{n\to\infty}\left(\frac{L_{n+2}}{L_{n+1}}\cdot\frac{L_{n+1}}{L_n}-\frac{L_{n+1}}{L_n}-1\right)=0\Leftrightarrow x^2-x-1=0,$$

$x=\lim_{n\to\infty}\frac{L_{n+1}}{L_n}$, so $x=\frac{1\pm\sqrt{5}}{2}$

Since $\frac{L_{n+1}}{L_n}>0, x=\lim_{n\to\infty}\frac{L_{n+1}}{L_n}=\frac{\sqrt{5}+1}{2}=\alpha.$

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!\,L_n}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{n!\,L_n}{n^n}}=\lim_{n\to\infty}\frac{(n+1)!\,L_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{n!\,L_n}=\lim_{n\to\infty}\frac{L_{n+1}}{L_n}\left(\frac{n}{n+1}\right)^n=\frac{\alpha}{e}$$

$$(1)\quad\sqrt[n+1]{(n+1)!\,L_{n+1}}-\sqrt[n]{n!\,L_n}=\sqrt[n]{n!\,L_n}(u_n-1)=$$

$$=\sqrt[n]{n!\,L_n}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n=$$

$=\frac{\sqrt[n]{n!L_n}}{n}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n, \forall n\geq 2$, where $u_n=\frac{\sqrt[n+1]{(n+1)!L_{n+1}}}{\sqrt[n]{n!L_n}}$.

$\lim_{n\to\infty}u_n=\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(n+1)!L_{n+1}}}{n+1}\cdot\frac{n}{\sqrt[n]{n!L_n}}\cdot\frac{n+1}{n}\right)=\frac{\alpha}{e}\cdot\frac{e}{\alpha}\cdot 1=1$, so $\lim_{n\to\infty}\frac{u_n-1}{\ln u_n}=1$;

$$\lim_{n\to\infty}u_n^n=\lim_{n\to\infty}\frac{(n+1)!\,L_{n+1}}{n!\,L_n}\cdot\frac{1}{\sqrt[n+1]{(n+1)!\,L_{n+1}}}=\lim_{n\to\infty}\frac{L_{n+1}}{L_n}\cdot\frac{n+1}{\sqrt[n+1]{(n+1)!\,L_{n+1}}}=$$

$$=\alpha\cdot\frac{e}{\alpha}=e$$

From (1): $\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!\,L_{n+1}}-\sqrt[n]{n!\,L_n}\right)=\frac{\alpha}{e}\cdot 1\cdot\ln\left(\lim_{n\to\infty}u_n^n\right)=\frac{\alpha}{e}\cdot\ln e=\frac{\alpha}{e}\cdot 1=\frac{\alpha}{e}$.

Analogously, $\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!\,F_{n+1}}-\sqrt[n]{n!\,F_n}\right)=\frac{\alpha}{e}$, where $\alpha=\frac{1+\sqrt{5}}{2}$.

By lemma 1, $\lim_{n\to\infty}I_n=\frac{1}{2}\lim_{n\to\infty}(y_{n+1}-y_n)=\frac{1}{2}\cdot\frac{\alpha}{e}=\frac{\alpha}{2e}$, where $\alpha=\frac{1+\sqrt{5}}{2}$ and we taking account by lemma 2, $\lim_{n\to\infty}(y_{n+1}-y_n)=\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!\,F_{n+1}}-\sqrt[n]{n!\,F_n}\right)=\frac{\alpha}{e}$