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UP.605 We consider the function $u: \mathbb{R} \rightarrow \mathbb{R}$, periodic with period 2π . For the period $[0, 2\pi]$ we have:

$u(x) = \cos(x)$ if $x \in [0, \frac{\pi}{2})$; $u(x) = 0$ if $x \in [\frac{\pi}{2}, \frac{3\pi}{2})$; $u(x) = \cos(x)$ if $x \in [\frac{3\pi}{2}, 2\pi)$. Prove the equality:

$$\int_0^{\infty} \frac{u(x)}{1+x^2} dx = \frac{\pi}{4e} + \frac{e^2+1}{2e} \arctan\left(\frac{1}{e}\right).$$

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Solution by proposer

Let us denote:

$$I = \int_0^{\infty} \frac{u(x)}{1+x^2} dx$$

The function $u(x)$ satisfies Dirichlet's conditions. Also, the function is even.

We expand the function in the Fourier series:

$$u(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos(nx)$$

where:

$$A_0 = \frac{1}{2\pi} \int_0^{2\pi} u(x) dx; A_n = \frac{1}{\pi} \int_0^{2\pi} u(x) \cos(nx) dx$$

Calculating these integrals, we obtain:

$$A_0 = \frac{1}{\pi}; A_1 = \frac{1}{2}; A_n = \frac{-1}{(n^2-1)\pi} \cdot \left[\cos\left(\frac{n\pi}{2}\right) + \cos\left(\frac{3n\pi}{2}\right) \right] \text{ for } n \geq 2$$

But:

$$\cos\left(\frac{3n\pi}{2}\right) = \cos\left(\frac{n\pi}{2}\right) \text{ for any } n \geq 2$$

Result:

$$A_n = -\frac{1}{n} \cdot \left[\frac{2 \cos\left(\frac{n\pi}{2}\right)}{n^2-1} \right] = \frac{-1}{\pi} \cdot \left[\frac{\cos\left(\frac{n\pi}{2}\right)}{n-1} - \frac{\cos\left(\frac{n\pi}{2}\right)}{n+1} \right] = B_n + C_n$$

Where

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$$B_n = \frac{-1}{\pi} \cdot \frac{\cos\left(\frac{n\pi}{2}\right)}{n-1}; \quad C_n = \frac{1}{\pi} \cdot \frac{\cos\left(\frac{n\pi}{2}\right)}{n+1}$$

We have:

$$B_n = 0 \text{ if } n = 3, 5, 7, \dots; B_n \neq 0 \text{ if } n = 2, 4, 6, \dots;$$

$$C_n = 0 \text{ if } n = 3, 5, 7, \dots; C_n \neq 0 \text{ if } n = 2, 4, 6, \dots;$$

So:

$$\begin{aligned} u(x) &= \frac{1}{\pi} + \frac{1}{2} \cos(x) + [B_2 \cos(2x) + B_4 \cos(4x) + B_6 \cos(6x) + B_8 \cos(8x) + \dots] + \\ &\quad + [C_2 \cos(2x) + C_4 \cos(4x) + C_6 \cos(6x) + C_8 \cos(8x) + \dots] \\ u(x) &= \frac{1}{\pi} + \frac{1}{2} \cos(x) + \left[\frac{1}{\pi} \cos(2x) - \frac{1}{3\pi} \cos(4x) + \frac{1}{5\pi} \cos(6x) - \frac{1}{7\pi} \cos(8x) + \dots \right] + \\ &\quad + \left[-\frac{1}{3\pi} \cos(2x) + \frac{1}{5\pi} \cos(4x) - \frac{1}{7\pi} \cos(6x) + \frac{1}{9\pi} \cos(8x) + \dots \right] \end{aligned}$$

We now use the following relationship:

$$\int_0^\infty \frac{\cos(mx)}{1+x^2} dx = \frac{\pi}{2} e^{-m}, \text{ where } m > 0$$

This relation is Laplace's integral and is well known.

It is easily proved for example using the properties of the Laplace transform.

We obtained the value of the integral I :

$$\begin{aligned} I &= \frac{1}{\pi} \frac{\pi}{2} + \frac{1}{2} \frac{\pi}{2} e^{-1} + \left(\frac{1}{\pi} \frac{\pi}{2} e^{-2} - \frac{1}{3\pi} \frac{\pi}{2} e^{-4} + \frac{1}{5\pi} \frac{\pi}{2} e^{-6} - \frac{1}{7\pi} \frac{\pi}{2} e^{-8} + \dots \right) + \\ &\quad + \left(-\frac{1}{3\pi} \frac{\pi}{2} e^{-2} + \frac{1}{5\pi} \frac{\pi}{2} e^{-4} - \frac{1}{7\pi} \frac{\pi}{2} e^{-6} + \frac{1}{9\pi} \frac{\pi}{2} e^{-8} + \dots \right) \\ 2I &= 1 + \frac{\pi}{2} e^{-1} + \left(e^{-2} - \frac{1}{3} e^{-4} + \frac{1}{5} e^{-6} - \frac{1}{7} e^{-8} + \dots \right) + \\ &\quad + \left(-\frac{1}{3} e^{-2} + \frac{1}{5} e^{-4} - \frac{1}{7} e^{-6} + \frac{1}{9} e^{-8} + \dots \right) \end{aligned}$$

We will now use the power series development of the following functions.

$$\arctan(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots \text{ where } -1 \leq x \leq 1$$

$$x \arctan(x) = x^2 - \frac{1}{3}x^4 + \frac{1}{5}x^6 - \frac{1}{7}x^8 + \dots \text{ where } -1 \leq x \leq 1$$

$$\frac{\arctan(x)}{x} = 1 - \frac{1}{3}x^2 + \frac{1}{5}x^4 - \frac{1}{7}x^6 + \dots \text{ where } -1 \leq x \leq 1$$

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We have:

$$e^{-1} \arctan(e^{-1}) = e^{-2} - \frac{1}{3}e^{-4} + \frac{1}{5}e^{-6} - \frac{1}{7}e^{-8} + \dots$$

$$\frac{\arctan(e^{-1})}{e^{-1}} = 1 - \frac{1}{3}e^{-2} + \frac{1}{5}e^{-4} - \frac{1}{7}e^{-6} + \dots$$

We obtain:

$$2I = \frac{\pi}{2}e^{-1} + e^{-1} \arctan(e^{-1}) + \frac{\arctan(e^{-1})}{e^{-1}}$$

Or:

$$I = \frac{\pi}{4e} + \frac{e^2 + 1}{2e} \arctan\left(\frac{1}{e}\right)$$

Thus, the problem is solved.