ROMANIAN MATHEMATICAL MAGAZINE

UP.604 Prove that $\frac{3}{2}$ is the smallest positive value of the power k such that

$$\frac{1}{a^k}+\frac{1}{b^k}+\frac{1}{c^k}\ge a^2+b^2+c^2$$

for all positive real numbers a,b,c with at most one of them smaller than 1 and $ab+bc+ca=3$.

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Solution by proposer

For $a\ge 1, b=1$ and $c\in(0,1]$ such that $a+c+ac=3$, the constraints are satisfied and the inequality becomes $E\ge 0$, where

$$E=\frac{1}{a^k}+\frac{1}{c^k}-a^2-c^2.$$

Assume that a and E are functions of c. For $c=1$, we have $a=1$. By differentiating the equality constraint, we get

$$(1+c)a'+1+a=0,\quad a'(1)=-1,$$
$$(1+c)a''+2a'=0,\quad a''(1)=1.$$

Therefore,

$$E'(c)=-\left(\frac{k}{a^{k+1}}+2a\right)a'-\frac{k}{c^{k+1}}-2c,\ \ E'(1)=0$$
$$E''(c)=-\left(\frac{k}{a^{k+1}}+2a\right)a''+\left(\frac{k(k+1)}{a^{k+2}}-2\right)(a')^2+\frac{k(k+1)}{c^{k+2}}-2,$$
$$E''(1)=(k+2)(2k-3)$$

Since $E(1)=E'(1)=0$, the condition $E''(1)\ge 0$ is necessary to have $E(c)\ge 0$ for $c\in(0,1]$. From this condition, we get $k\ge\frac{3}{2}$. To show that $\frac{3}{2}$ is the smallest positive value of k, we need to prove that $F\ge 0$ for $a\ge b\ge 1\ge c$ and $ab+bc+ca=3$, where

$$F=a^{-\frac{3}{2}}+b^{-\frac{3}{2}}+c^{-\frac{3}{2}}-a^2-b^2-c^2.$$

For fixed c, assume that a and F are functions of b. We have

$$(b+c)a'+a+c=0$$

and

$$F'(b) = -\left(\frac{3a^{-\frac{5}{2}}}{2} + 2a\right)a' - \frac{3b^{-\frac{5}{2}}}{2} - 2b = \left(\frac{3a^{-\frac{5}{2}}}{2} + 2a\right)\frac{a+c}{b+c} - \frac{3b^{-\frac{5}{2}}}{2} - 2b.$$

We claim that $F'(b) \geq 0$, that is

$$\frac{a+c}{b+c} \geq \frac{3b^{-\frac{5}{2}} + 4b}{3a^{-\frac{5}{2}} + 4a}, \qquad \frac{a+c}{b+c} - 1 \geq \frac{3b^{-\frac{5}{2}} + 4b}{3a^{-\frac{5}{2}} + 4a} - 1,$$

$$\frac{a-b}{b+c} \geq \frac{3\left(b^{-\frac{5}{2}} - a^{-\frac{5}{2}}\right) - 4(a-b)}{3a^{-\frac{5}{2}} + 4a}, \quad \frac{a-b}{b+c} \geq \frac{3\left(a^{\frac{5}{2}} - b^{\frac{5}{2}}\right) - 4a^{\frac{5}{2}}b^{\frac{5}{2}}(a-b)}{b^{\frac{5}{2}}\left(4a^{\frac{7}{2}} + 3\right)}.$$

Denoting $x = a^{\frac{1}{2}}$ and $y = b^{\frac{1}{2}}$, we need to show that if $x \geq y \geq 1$, then

$$\frac{x^2 - y^2}{y^2 + c} \geq \frac{3\left(x^5 - y^5\right) - 4x^5y^5(x^2 - y^2)}{y^5(4x^7 + 3)}.$$

It is true if

$$\frac{x+y}{y^2 + c} \geq \frac{3(x^4 + x^3y + x^2y^2 + xy^3 + y^4) - 4x^5y^5(x+y)}{y^5(4x^7 + 3)}.$$

Since

$$\frac{x+y}{y^2+c} \geq \frac{x+y}{y^2+1} \geq \frac{2}{y^2+1} \geq \frac{1}{y^2} \geq \frac{1}{y^5},$$

it suffices to show that

$$1 \geq \frac{3(x^4 + x^3y + x^2y^2 + xy^3 + y^4) - 4x^5y^5(x+y)}{4x^7 + 3},$$

that is

$$4x^7 + 3 + 4x^5y^5(x+y) \geq 3(x^4 + x^3y + x^2y^2 + xy^3 + y^4).$$

It is true if

$$4x^7 + 3 + 4x^5(x+1) \geq 15x^4,$$

that is

$$4x^7 + 4x^6 + 4x^5 - 15x^4 + 3 \geq 0.$$

Indeed, we have

$$4x^7 + 4x^6 + 4x^5 - 15x^4 + 3 \geq 4x^6 + 4x^6 + 4x^4 - 15x^4 + 3 = 8x^6 - 11x^5 + 3$$
$$= (x^2 - 1)(8x^4 - 3x^2 - 3) \geq 0$$

From $F'(b) \geq 0$, it follows that $F(b)$ is increasing and has the minimum value when b is minimum, hence when $b = 1$. So, it suffices to consider the case $b = 1$, when we need to show that

$$\frac{1}{a\sqrt{a}} + \frac{1}{c\sqrt{c}} \geq a^2 + c^2$$

for $a + c + ac = 3$. Denoting $ac = t$, we have $a + c = 3 - t$. From $3 - t = a + c \geq 2\sqrt{t}$, we get $t \leq 1$. By squaring, the required inequality becomes as follows

$$\left(a\sqrt{a} + c\sqrt{c}\right)^2 \geq a^3c^3(a^2 + c^2)^2.$$

By Bernoulli's inequality, we have

$$t^{\frac{3}{2}} = [1 + (t - 1)]^{\frac{3}{2}} \geq 1 + \frac{3(t - 1)}{2} = \frac{3t - 1}{2}, \qquad 2t^{\frac{3}{2}} \geq 3t - 1.$$

Since $a^2 + c^2 = (a + c)^2 - 2ac = (3 - t)^2 - 2t = 9 - 8t + t^2$ and

$$\left(a\sqrt{a} + c\sqrt{c}\right)^2 = (a + c)^3 - 3ac(a + c) + 2ac\sqrt{ac} \geq (3 - t)^3 - 3t(3 - t) + (3t - 1)$$
$$= 26 - 33t + 12t^2 - t^3,$$

it suffices to show that

$$26 - 33t + 12t^2 - t^3 \geq t^3(9 - 8t + t^2)^2,$$

that is

$$(1 - t)^2 f(t) \geq 0, f(t) = 26 + 19t + 24t^2 - 53t^3 + 14t^4 - t^5.$$

It is true since

$$f(t) > 10 + 19t + 24t^2 - 53t^3 = (1 - t)(10 + 29t + 53t^2) \geq 0.$$

If $k = \frac{3}{2}$, then the equality occurs for $a = b = c = 1$.