ROMANIAN MATHEMATICAL MAGAZINE

UP.603 Find all positive real values of the constant k such that

$$9(a^2+k)(b^2+k)(c^2+k) \le (1+k)^3(a+b+c)^2$$

for any nonnegative real numbers a,b,c with $ab+bc+ca=3$.

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Solution by proposer

For $a=0$ and $b=c=\sqrt{3}$, the equality constraint is satisfied and the inequality becomes $f(k)\ge 0$, where

$$f(k)=k^3-6k^2-15k+4$$

So,

$$k\in \mathbb{I}=(0,k_1]\cup[k_2,\infty),$$

where $k_1\approx 0.2438$ and $k_2\approx 7.8467$ are the positive roots of the equation $f(k)=0$. To show that the range of values of k is $\mathbb{I}$, we need to prove the required inequality for $k\in\mathbb{I}$, that is for $k>0$ such that $f(k)\ge 0$. Denote $p=a+b+c$ and $r=abc$. For fixed p, we write the inequality as $F(r)\ge 0$, where

$$F(r)=-r^2+2kpr+C(p).$$

Since $F(r)$ is a quadratic concave function, it suffices to consider the cases when r has the minimum and maximum values. As it is known, the minimum and maximum values of r for fixed p and q are achieved when one of a,b,c is 0 or two of a,b,c are equal. So, it suffices to consider these cases.

Case 1: $a=0$. We need to prove that $bc=3$ implies $G\ge 0$, where

$$G=(1+k)^3(b+c)^2-9k(b^2+k)(c^2+k).$$

Denoting $S=b+c$, we have $S^2\ge 4bc=12$ and

$$G=(1+k)^3S^2-9k(kS^2+k^2-6k+9)=AS^2-9k(k^2-6k+9),$$

where $A=k^3-6k^2+3k=1$. Since

$$4A-f(k)=4(k^3-6k^2+3k+1)-(k^3-6k^2-15k+4)=3k(k-3)^2>0,$$

we have $4A>f(k)\ge 0$ and

$$G\ge 12A-9k(k^2-6k+9)=3f(k)\ge 0.$$

Case 2: $b=c$. Denote $b=c\coloneqq x$. From the equality constraint, we get $2ax+x^2=3$, with $x^2\in(0,3]$. The required inequality becomes as follows:

$$[(3-x^2)^2+4kx^2](x^2+k)^2 \le (1+k)^3(x^2+1)^2,$$

$$(x^2-1)^2[x^4+(6k-4)x^2-k^3+6k^2-3k-1] \le 0$$

Let $t=x^2, t \in (0,3]$. The inequality is true if $q(t)<0$, where

$$g(t)=t^2+(6k-4)t-k^3+6k^2-3k-1.$$

Since $g(t)$ is a quadratic convex function, it suffices to show that $g(0) \le 0$ and $g(3) \le 0$.

Indeed, $g(0)=-k^3+6k^2-3k-1=-A<0$ (see Case 1), and $g(3)=-f(k) \le 0$.

For $k \in (0,k_1] \cup [k_2,\infty)$, the equality occurs when $a=b=c=1$. For $k=k_1$ and $k=k_2$, the equality also occurs when $a=0$ and $b=c=\sqrt{3}$ (or any cyclic permutation).