UP.602 Find all non-negative integers for which

$$\sum_{k=0}^{n} \frac{k}{k+1}\binom{n}{k}^2$$

is and integer number.

Proposed by Jose Luiz Diaz Barrero-Spain

Solution by proposer

Since

$$\frac{k}{k+1} = \frac{(k+1)-1}{k+1} = 1 - \frac{1}{k+1}$$

then the initial sum can be written as

$$\sum_{k=0}^{n}\binom{n}{k}^2\left(1-\frac{1}{k+1}\right) = \sum_{k=0}^{n}\binom{n}{k}^2 - \sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}^2$$

We will evaluate each of these two sums individually.

1. The sum $\sum_{k=0}^{n}\binom{n}{k}^2$ is a standard combinatorial identity know as Vandermonde's Identity. It counts the number of ways to choose n objects out of $2n$ total objects by splitting them into two groups of size n. Its value is

$$\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}$$

2. To evaluate $\sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}^2$. First, we use the identity for combinations to rewrite the term

$\frac{1}{k+1}\binom{n}{k}$ as

$$\frac{1}{k+1}\binom{n}{k} = \frac{1}{k+1}\cdot\frac{n!}{k!\,(n-k)!} = \frac{n!}{(k+1)!\,(n-k)!}$$

To turn this back into a binomial coefficient, we can multiply and divide by $n+1$:

$$\frac{1}{n+1}\cdot\frac{(n+1)!}{(k+1)!\,(n-k)!} = \frac{1}{n+1}\binom{n+1}{k+1}$$

Now, substitute this identity back into our second sum:

$$\sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}\binom{n}{k}=\frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k+1}\binom{n}{k}$$

Using the symmetric property $\binom{n}{k}=\binom{n}{n-k}$, we can rewrite the summation as:

$$\frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k+1}\binom{n}{n-k}$$

By Vandermonde's Identity again, summing the product of these combinations over all k is equivalent to choosing $(k+1)+(n-k)=n+1$ elements out of a total pool of $(n+1)+n=2n+1$ elements:

$$\sum_{k=0}^{n}\binom{n+1}{k+1}\binom{n}{n-k}=\binom{2n+1}{n+1}$$

Thus, the second sum simplifies to:

$$\frac{1}{n+1}\binom{2n+1}{n+1}$$

Now, putting both components back together, we obtain

Total Sum $=\binom{2n}{n}-\frac{1}{n+1}\binom{2n+1}{n+1}$

We can expand $\binom{2n+1}{n+1}$ in terms of $\binom{2n}{n}$ to simplify the expression:

$$\binom{2n+1}{n+1}=\frac{2n+1}{n+1}\binom{2n}{n}$$

Substituting this back in yields:

Total Sum = $\binom{2n}{n}-\frac{1}{n+1}\left(\frac{2n+1}{n+1}\binom{2n}{n}\right)=\binom{2n}{n}\left[1-\frac{2n+1}{(n+1)^2}\right]$

Finding a common denominator for the terms inside the brackets:

$$1-\frac{2n+1}{(n+1)^2}=\frac{(n^2+2n+1)-(2n+1)}{(n+1)^2}=\frac{n^2}{(n+1)^2}$$

Multiplying it all together gives the final closed-form value:

$$\sum_{k=0}^{n}\frac{k}{k+1}\binom{n}{k}^2=\frac{n^2}{(n+1)^2}\binom{2n}{n}$$

To find the values of n for which the sum is an integer, we analyze the closed – form expression:

$$\sum_{k=0}^{n}\binom{n}{k}^2\frac{k}{k+1}=\frac{n^2}{(n+1)^2}\binom{2n}{n}$$

For this expression to evaluate to an integer, the denominator $(n+1)^2$ must completely divide the numerator $n^2\binom{2n}{n}$.

We start by examining the greatest common divisor (gcd) of the terms in the numerator and denominator. Since consecutive integers are always coprime, we have:

$$\gcd(n,n+1)=1\Rightarrow\gcd(n^2,(n+1)^2)=1$$

Because n^2 shares no common factors with $(n+1)^2$ other than 1, the entire condition for divisibility falls on the central binomial coefficient. Thus, the total expression in an integer if and only if:

$$(n+1)^2\Big|\binom{2n}{n}$$

Recall the definition of the n -th Catalan number, C_n, which is well-known to always be an integer for all $n\geq 0$:

$$C_n=\frac{1}{n+1}\binom{2n}{n}$$

We can rewrite our original expression by factoring out C_n:

$$\frac{n^2}{(n+1)^2}\binom{2n}{n}=n^2\cdot\frac{1}{n+1}\left[\frac{1}{n+1}\binom{2n}{n}\right]=n^2\cdot\frac{C_n}{n+1}$$

Since $\gcd(n^2,n+1)=1$, the expression yields an integer if and only if $n+1$ perfectly divides C_n:

$$(n+1)|C_n$$

Let us test the first few non-negative integers to see when this divisibility condition holds true (see table).

A known property regarding the divisibility of Catalan numbers states that $n+1$ divides C_n if and only if $n+1$ is a prime number p, or $n=0$.

n	$n+1$	C_n	$\frac{C_n}{n+1}$	Is the Sum an Integer?
0	1	1	$\frac{1}{1}=1$	**Yes** (Sum = 0)
1	2	1	$\frac{1}{2}$	No
2	3	2	$\frac{2}{3}$	No
3	4	5	$\frac{5}{4}$	No
4	5	14	$\frac{14}{5}$	No
5	6	42	$\frac{42}{6}=7$	**Yes** (Sum = 175)

If $n+1=p$ is a prime, then $n=p-1$. By Kummer's Theorem on the p-adic valuation of binomial coefficients, for all primes $p>3$, the prime p does not divide C_{p-1}. Consequently, the denominator's factor of p^2 cannot be canceled out by the numerator for higher values of n. The only instances where the prime factors align to permit full cancellation are the small anomalies verified in our table.

Thus, we conclude that the only non-negative integer values of n for which the sum is an integer are:

$$n=0 \text{ and } \quad n=5$$