ROMANIAN MATHEMATICAL MAGAZINE

UP.601 Let $\{L_n\}_{n\geq 0}$ be the Lucas' sequence defined by $L_0 = 2, L_1 = 1$ and for all $n \geq 2, L_n = L_{n-1} + L_{n-2}$. Determine the sum of the lengths of the intervals, disjoint two by two, formed by all $f(x) = 1$, where

$$f(x) = \frac{L_1^2}{x + L_1^2} + \frac{L_2^2}{x + L_2^2} + \cdots + \frac{L_n^2}{x + L_n^2}$$

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Solution by proposer

To find the sum of the lengths of the intervals where $f(x) > 1$, we analyze the behavior of the function and find its roots using Vieta's formulas. The function is given by:

$$f(x) = \frac{L_1^2}{x + L_1^2} + \frac{L_2^2}{x + L_2^2} + \cdots + \frac{L_n^2}{x + L_n^2}$$

The poles (vertical asymptotes) of $f(x)$ occur at $x = -L_i^2$. Since the Lucas numbers grow strictly monotonically for $n \geq 1$, we can arrange these poles as:

$$-L_n^2 < -L_{n-1}^2 < \cdots < -L_2^2 < -L_1^2 < 0$$

Taking the derivative of $f(x)$ shows that it is strictly decreasing on every interval where it is defined:

$$f'(x) = -\sum_{i=1}^{n} \frac{L_i^2}{\left(x + L_i^2\right)^2} < 0$$

As x approaches any pole $-L_k^2$ from the right, $f(x) \to +\infty$. As it approaches from the left, $f(x) \to -\infty$. Thus, the equation $f(x) = 1$ has exactly n real roots, denoted as $x_0, x_1, \dots, x_{n-1}$, situated in the following intervals:

$$x_0 \in \left(-L_1^2, \infty\right), x_1 \in \left(-L_2^2, -L_1^2\right), \dots, x_{n-1} \in \left(-L_n^2, -L_{n-1}^2\right)$$

Consequently, the disjoint intervals where $f(x) \geq 1$ are:

$$\left(-L_1^2, x_0\right], \left(-L_2^2, x_1\right], \dots, \left(-L_n^2, x_{n-1}\right]$$

The sum of the lengths of these intervals is:

Total Length $= \left(x_0 + L_1^2\right) + \left(x_1 + L_2^2\right) + \cdots + \left(x_{n-1} + L_n^2\right) = \sum_{k=0}^{n-1} x_k + \sum_{i=1}^{n} L_i^2$

To find $\sum_{k=0}^{n-1} x_k$, we look at the polynomial form of $f(x) = 1$:

$$\sum_{i=1}^{n} L_i^2 \prod_{j\neq i}(x+L_j^2)=\prod_{j=1}^{n}(x+L_j^2)$$

Expanding both sides to find the coefficients of x^n and x^{n-1}:

- **RHS:** $x^n+\left(\sum_{i=1}^{n} L_i^2\right)x^{n-1}+\cdots$
- **LHS:** $\left(\sum_{i=1}^{n} L_i^2\right)x^{n-1}+\cdots$

When subtracting the LHS from the RHS, we obtain

$$x^n+0\cdot x^{n-1}+\cdots=0$$

By Vieta's formulas, the sum of the roots is equal to the negative of the coefficient of x^{n-1}, meaning $\sum_{k=0}^{n-1} x_k=0$. Therefore, the total length simplifies to:

Total Length $=\sum_{i=1}^{n} L_i^2$

Using the recurrence relation for Lucas numbers, $L_k=L_{k+1}-L_{k-1}$, we rewrite each squared term to create a telescoping sum:

$$L_k^2=L_k(L_{k+1}-L_k-1)=L_kL_{k+1}-L_{k-1}L_k$$

Summing this from 1 to n, we obtain

$$\sum_{i=1}^{n} L_i^2=(L_1L_2-L_0L_1)+(L_2L_3-L_1L_2)+\cdots+(L_nL_{n+1}-L_{n-1}L_n)$$

$$=L_nL_{n+1}-L_0L_1=L_nL_{n+1}-2.$$

Thus, the sum of the lengths of the intervals is $L_nL_{n+1}-2$.