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U.3450. *Proposed by D.M. Bătineţu-Giurgiu, Mihaly Bencze, Romania.* Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be sequences of real strictly positive numbers such that

$$\lim_{n\to\infty} \frac{a_{n+1}}{n\cdot a_n} = a > 0, \quad \lim_{n\to\infty} \frac{b_{n+1}}{n\cdot b_n} = b > 0$$

Find

$$\lim_{n\to\infty} \frac{1}{\sqrt[n]{a_n}} \cdot \sum_{k=1}^{n} \frac{k}{\sqrt[n]{b_k}}$$

Solution by José Luis Díaz Barrero, Barcelona, Spain. To find the limit:

$$\lim_{n\to\infty} \frac{1}{\sqrt[n]{a_n}} \cdot \sum_{k=1}^{n} \frac{k}{\sqrt[n]{b_k}}$$

we analyze the asymptotic behavior of $\sqrt[n]{a_n}$ and $\sqrt[n]{b_k}$ using the given ratio conditions.

1. Analysis of the Sequence (a_n): We are given

$$\lim_{n\to\infty} \frac{a_{n+1}}{n\cdot a_n} = a > 0$$

A sequence satisfying this growth condition behaves proportionally to $n!\cdot a^n$. Specifically, if $a_n \sim \frac{n!}{e^n}a^n$, then the ratio $\frac{a_{n+1}}{n\cdot a_n}$ evaluates to a and the n-th root $\sqrt[n]{a_n}$ behaves according to Stirling's approximation, where $\sqrt[n]{n!} \sim \frac{n}{e}$.

Thus, we obtain $\lim\limits_{n\to\infty} \dfrac{\sqrt[n]{a_n}}{n} = \dfrac{a}{e}$. From this limit, we can express the asymptotic behavior as

$$\sqrt[n]{a_n} \sim \frac{a}{e}\cdot n \Rightarrow \frac{1}{\sqrt[n]{a_n}} \sim \frac{e}{a\cdot n}$$

2. Analysis of the Sequence (b_n): Similarly, for the sequence (b_n) with $\lim_{n\to\infty} \frac{b_{n+1}}{n\cdot b_n} = b > 0$, the corresponding root behavior yields:

$$\sqrt[n]{b_k} \sim \frac{b}{e}\cdot k$$

Substituting this approximation into the summation term gives:

$$\sum_{k=1}^{n} \frac{k}{\sqrt[n]{b_k}} \approx \sum_{k=1}^{n} \frac{k}{\frac{b}{e}\cdot k} = \sum_{k=1}^{n} \frac{e}{b} = \frac{e}{b}\cdot n$$

3. Evaluation of the Limit: Substituting our asymptotic expressions back into the original product:

$$\lim_{n\to\infty} \left(\frac{1}{\sqrt[n]{a_n}} \cdot \sum_{k=1}^{n} \frac{k}{\sqrt[n]{b_k}} \right) = \lim_{n\to\infty} \left(\left(\frac{e}{a\cdot n}\right) \cdot \left(\frac{e}{b}\cdot n\right) \right) = \frac{e^2}{ab}$$