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U.3449. *Proposed by D.M. Bătineţu-Giurgiu, Romania.* Let $(a_n)_{n\geq 1}$ be a sequence of strictly positive real numbers, $a_n > 0$ for all $n \in \mathbb{N}^*$, such that

$$\lim_{n\to\infty} \frac{a_{n+1}}{n \cdot a_n} = a > 0$$

Find

$$\lim_{n\to\infty} n \cdot \sqrt[n]{a_n} \cdot \int_0^1 x^3 \cdot (1-x^2)^n \, dx$$

Solution by José Luis Díaz Barrero, Barcelona, Spain. To find the limit we evaluate the sequence component and the definite integral separately.

We are given

$$\lim_{n\to\infty} \frac{a_{n+1}}{n \cdot a_n} = a > 0$$

Using the relation between ratio limits and root limits (via Stirling's approximation where $\sqrt[n]{n!} \sim \frac{n}{e}$), we obtain

$$\lim_{n\to\infty} \frac{\sqrt[n]{a_n}}{n} = \frac{a}{e} \Rightarrow \sqrt[n]{a_n} \sim \frac{a}{e} \cdot n$$

Multiplying by n gives the asymptotic behavior for our expression

$$n \cdot \sqrt[n]{a_n} \approx n \cdot \left(\frac{a}{e} \cdot n\right) = \frac{a}{e} \cdot n^2$$

Let the integral be

$$I_n = \int_0^1 x^3(1-x^2)^n \, dx$$

Using the substitution $u = 1 - x^2$, which implies $du = -2x \, dx$ (or $x \, dx = -\frac{1}{2} \, du$) and $x^2 = 1 - u$, we have

$$I_n = \int_1^0 (1-u) \cdot u^n \cdot \left(-\frac{1}{2} \, du\right) = \frac{1}{2} \int_0^1 (1-u)u^n \, du$$

$$I_n = \frac{1}{2} \int_0^1 (u^n - u^{n+1}) \, du = \frac{1}{2} \left[\frac{u^{n+1}}{n+1} - \frac{u^{n+2}}{n+2}\right]_0^1$$

$$I_n = \frac{1}{2} \left(\frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{1}{2(n+1)(n+2)}$$

Substituting these expressions back into the original product yields

$$\lim_{n\to\infty} \left(n \cdot \sqrt[n]{a_n} \cdot \int_0^1 x^3(1-x^2)^n \, dx\right) = \lim_{n\to\infty} \left(\left(\frac{a}{e} \cdot n^2\right) \cdot \left(\frac{1}{2(n+1)(n+2)}\right)\right) = \frac{a}{2e}$$