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U.3447. *Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru, Romania.* If $t \geq 0$, find

$$\lim_{n\to\infty}\left(\frac{\left(\sqrt[n+1]{(n+1)!}\right)^{t+1}}{(n+1)^t}-\frac{\left(\sqrt[n]{n!}\right)^{t+1}}{n^t}\right)$$

Solution by José Luis Díaz Barrero, Barcelona, Spain. To evaluate the limit for $t \geq 0$, we define a function $f(n)$ by

$$f(n)=\frac{\left(\sqrt[n]{n!}\right)^{t+1}}{n^t}$$

The expression inside the limit is then the difference $f(n+1)-f(n)$.

Recall from standard limits involving factorials (via Stirling's approximation) that

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\frac{1}{e}\Rightarrow\sqrt[n]{n!}\sim\frac{n}{e}$$

Substituting this asymptotic approximation into $f(n)$ yields

$$f(n)=\frac{\left(\sqrt[n]{n!}\right)^{t+1}}{n^t}\sim\frac{\left(\frac{n}{e}\right)^{t+1}}{n^t}=\frac{n}{e^{t+1}}$$

Now, look at the difference between consecutive terms

$$f(n+1)-f(n)\sim\frac{n+1}{e^{t+1}}-\frac{n}{e^{t+1}}$$

Factor out the constant $\frac{1}{e^{t+1}}$ to obtain

$$f(n+1)-f(n)\sim\frac{(n+1)-n}{e^{t+1}}=\frac{1}{e^{t+1}}$$

Finally, we conclude that

$$\lim_{n\to\infty}\left(\frac{\left(\sqrt[n+1]{(n+1)!}\right)^{t+1}}{(n+1)^t}-\frac{\left(\sqrt[n]{n!}\right)^{t+1}}{n^t}\right)=\frac{1}{e^{t+1}}$$