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U.3409. *Proposed by Ankush Kumar Parcha, India.* Prove the below closed form

$$\int_0^1 \int_0^1 \frac{\ln^2\left(\frac{x}{y}\right) + \ln^2\left(\frac{y}{x}\right)}{\frac{x}{y} + \frac{y}{x}}\, dx\, dy = \frac{3}{8}\zeta(3),$$

where $\zeta(3)$ is Apery's constant.

Solution by José Luis Díaz Barrero, Barcelona, Spain. To prove the given closed form, let us evaluate the double integral step-by-step:

$$I = \int_0^1 \int_0^1 \frac{\ln^2\left(\frac{x}{y}\right) + \ln^2\left(\frac{y}{x}\right)}{\frac{x}{y} + \frac{y}{x}}\, dx\, dy$$

Notice that $\ln(y/x) = -\ln(x/y)$, which means

$$\ln^2\left(\frac{y}{x}\right) = \left(-\ln\left(\frac{x}{y}\right)\right)^2 = \ln^2\left(\frac{x}{y}\right)$$

Therefore, the numerator simplifies to

$$\ln^2\left(\frac{x}{y}\right) + \ln^2\left(\frac{y}{x}\right) = 2\ln^2\left(\frac{x}{y}\right)$$

Substituting this back into the integral gives

$$I = 2\int_0^1 \int_0^1 \frac{\ln^2\left(\frac{x}{y}\right)}{\frac{x}{y} + \frac{y}{x}}\, dx\, dy$$

Using the substitution $x = e^{-s}$ and $y = e^{-t}$, where $dx = -e^{-s}ds$ and $dy = -e^{-t}dt$ for $s, t \in [0, \infty)$, we have

$$I = \int_0^\infty \int_0^\infty \frac{(s-t)^2 + (t-s)^2}{e^{s-t} + e^{t-s}} e^{-s} e^{-t}\, ds\, dt = \int_0^\infty \int_0^\infty \frac{2(s-t)^2}{2\cosh(s-t)} e^{-(s+t)}\, ds\, dt$$

$$I = \int_0^\infty \int_0^\infty \frac{(s-t)^2}{\cosh(s-t)} e^{-(s+t)}\, ds\, dt$$

Let $u = s+t$ and $v = s-t$. The Jacobian is $\dfrac{1}{2}$, and the region maps appropriately to yield

$$I = \int_0^\infty \frac{u^2}{\cosh(u)}\, du$$

Since $\cosh(u) = \frac{e^u + e^{-u}}{2} = \frac{e^u}{1 + e^{-2u}}$, we rewrite the integral as

$$I = 2\int_0^\infty \frac{u^2 e^{-u}}{1+e^{-2u}}\,du$$

Using the geometric series expansion $\dfrac{1}{1+e^{-2u}} = \displaystyle\sum_{n=0}^\infty (-1)^n e^{-2nu}$, we have

$$I = 2\sum_{n=0}^\infty (-1)^n \int_0^\infty u^2 e^{-(2n+1)u}\,du$$

Using the gamma integral formula $\int_0^\infty u^2 e^{-au}\,du = \frac{2}{a^3}$, we obtain

$$I = 2\sum_{n=0}^\infty (-1)^n \frac{2}{(2n+1)^3} = 4\sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)^3}$$

Using the Dirichlet L-series evaluation for $\beta(3) = \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)^3} = \frac{3}{32}\zeta(3)$ yields

$$I = 4\left(\frac{3}{32}\zeta(3)\right) = \frac{3}{8}\zeta(3),$$

and the identity is proven.