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U.3404. *Proposed by Vasile Mircea Popa, Romania.* Find

$$\Omega = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{5^{\frac{k}{n}}}{n + \frac{1}{k}}$$

Solution by José Luis Díaz Barrero, Barcelona, Spain. Factor out n from the denominator of each term to get

$$\frac{5^{\frac{k}{n}}}{n + \frac{1}{k}} = \frac{1}{n} \cdot \frac{5^{\frac{k}{n}}}{1 + \frac{1}{nk}}$$

Thus, the sum becomes

$$\Omega = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n} \frac{5^{\frac{k}{n}}}{1 + \frac{1}{nk}}$$

Recall the definition of the definite integral as a Riemann sum for a continuous function $f(x)$ on $[0,1]$ with uniform partition width $\Delta x = \frac{1}{n}$:

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right)$$

For $f(x) = 5^x$, the standard Riemann sum is

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} 5^{\frac{k}{n}} = \int_0^1 5^x\,dx$$

Let's check the difference between our expression and the standard Riemann sum

$$\text{Difference} = \frac{1}{n} \sum_{k=1}^{n} \left(\frac{5^{\frac{k}{n}}}{1 + \frac{1}{nk}} - 5^{\frac{k}{n}} \right) = -\frac{1}{n^2} \sum_{k=1}^{n} \frac{5^{\frac{k}{n}}}{k\left(1 + \frac{1}{nk}\right)}$$

Since $5^{\frac{k}{n}} \le 5$ and $1 + \frac{1}{nk} \ge 1$ for $k \ge 1$, we can bound the absolute difference as follows

$$|\text{Difference}| \le \frac{1}{n^2} \sum_{k=1}^{n} \frac{5}{k} = \frac{5}{n^2} \sum_{k=1}^{n} \frac{1}{k} = \frac{5H_n}{n^2}$$

where $H_n = \sum_{k=1}^{n} \frac{1}{k}$ is the n-th harmonic number, which grows logarithmically ($\approx \ln n$). As $n \to \infty$ then $\lim_{n\to\infty} \frac{5\ln n}{n^2} = 0$. Therefore, the difference vanishes in the limit, meaning our sum approaches the same limit as the standard Riemann sum.

Next, we evaluate the integral and we obtain

$$\Omega = \int_0^1 5^x\,dx = \left[\frac{5^x}{\ln 5}\right]_0^1 = \frac{4}{\ln 5}$$