José Luis **Díaz-Barrero**
Barcelona Mathematical Circle (BMC)
jose.luis.diaz@upc.edu

U.3380. *Proposed by Marin Chirciu, Romania.* Let $n \in \mathbb{N}$ fixed. Solve for real numbers

$$[x] + \frac{n}{[x]} = \{x\} + \frac{n}{\{x\}}$$

Solution by José Luis Díaz Barrero, Barcelona, Spain. Let x be a real number. We can express x using its floor (integer part) and fractional part as

$$x = [x] + \{x\}$$

where $[x] \in \mathbb{Z}$ and $0 \leq \{x\} < 1$.
Since division by zero is undefined, we must restrict our domain

1. $[x] \neq 0$
2. $\{x\} \neq 0$ (meaning x is not an integer)

Combining these constraints, the integer part $[x]$ cannot be 0, and $\{x\}$ cannot be 0.

Rewrite the original equation by grouping the terms involving the integer parts on one side and the fractional parts on the other to obtain

$$[x] - \{x\} = \frac{n}{\{x\}} - \frac{n}{[x]} = \frac{n[x] - n\{x\}}{[x]\{x\}} = \frac{n([x] - \{x\})}{[x]\{x\}}$$

Bringing all terms to one side yields

$$([x] - \{x\}) - \frac{n([x] - \{x\})}{[x]\{x\}} = 0 \Leftrightarrow ([x] - \{x\}) \left(1 - \frac{n}{[x]\{x\}}\right) = 0$$

This gives us two possible cases for the solution:

s
1. $[x] - \{x\} = 0$. This implies $[x] = \{x\}$. Since the fractional part must satisfy $0 \leq \{x\} < 1$, the integer part must also satisfy $0 \leq [x] < 1$. Because $[x]$ must be an integer, the only possibility is:

$$[x] = 0$$

However, we established previously that $[x] \neq 0$. Therefore, this case yields no valid solutions.

2. $1 - \frac{n}{[x]\{x\}} = 0$. This implies

$$\frac{n}{[x]\{x\}} = 1 \Rightarrow [x]\{x\} = n$$

Let us test the constraints for this case:

- Since $n \in \mathbb{N}$ $(n \geq 1)$, for $[x]\{x\} = n$ to hold, $[x]$ and $\{x\}$ must have the same sign.
- By definition, $\{x\} \geq 0$, which means $[x]$ must be positive $([x] > 0)$.
- Furthermore, since $0 \leq \{x\} < 1$, we have:

$$\{x\} = \frac{n}{[x]} < 1 \Rightarrow n < [x]$$

Since $[x]$ must be an integer strictly greater than n, let $[x] = k$, where k is an integer and $k > n$. Substituting $[x] = k$ into our relation for $\{x\}$ yields

$$\{x\} = \frac{n}{k}$$

Thus, the number x can be written as

$$x = [x] + \{x\} = k + \frac{n}{k},$$

where k is any integer such that $k > n$. This, allows us to conclude that the solutions are the real numbers x given by

$$x = k + \frac{n}{k}, \quad \text{where } k \in \mathbb{Z} \text{ and } k > n$$