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**U.3375.** *Proposed by Marin Chirciu, Romania.* Let $a > 0$ and $\lambda > 0$ fixed. Solve for real numbers

$$(ax+1)\left(1+\sqrt{(ax+1)^2+\lambda}\right)+x\left(1+\sqrt{x^2+\lambda}\right)=0$$

*Solution by José Luis Díaz Barrero, Barcelona, Spain.* Let us define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(t) = t\left(1+\sqrt{t^2+\lambda}\right)$$

Using this definition, the original equation can be rewritten as

$$f(ax+1)+f(x)=0$$

which is equivalent to

$$f(ax+1)=-f(x)$$

Now, we examine the behavior and symmetry of $f(t)$:

1. Strict Monotonicity: Take the derivative of $f(t)$ with respect to $t$ is

   $$f'(t)=1+\sqrt{t^2+\lambda}+\frac{t^2}{\sqrt{t^2+\lambda}}$$

   Since $\lambda > 0$, the term $\sqrt{t^2+\lambda}$ is always strictly positive for all real $t$. Thus, $f'(t) > 0$ for all $t \in \mathbb{R}$. This means $f(t)$ is a strictly increasing function on the entire real line and is therefore one-to-one (injective).

2. Odd Symmetry: Let us evaluate $f(-t)$:

   $$f(-t)=(-t)\left(1+\sqrt{(-t)^2+\lambda}\right)=-t\left(1+\sqrt{t^2+\lambda}\right)=-f(t)$$

   Thus, $f(t)$ is an odd function.

Using the odd property of $f$, say $-f(x) = f(-x)$, we can rewrite the equation $f(ax+1) = -f(x)$ as $f(ax+1) = f(-x)$. Since $f$ is strictly monotonic (one-to-one), we can equate its arguments directly $ax+1 = -x$ from which we obtain

$$x=-\frac{1}{a+1}$$

Since $a > 0$, the denominator $a+1$ is never zero, and the unique real solution to the equation is

$$x=-\frac{1}{a+1}$$