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U.3370. *Proposed by Marin Chirciu, Romania.* Let $a \geq 1$. If $x > 0$ then prove that

$$\frac{1}{\sqrt{x+a}} + \frac{x}{2\sqrt{ax^2+1}} \leq \frac{3}{2\sqrt{a+1}}$$

Solution by José Luis Díaz Barrero, Barcelona, Spain. Consider the function $f : (0, +\infty) \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{\sqrt{x+a}} + \frac{x}{2\sqrt{ax^2+1}}$$

Evaluating $f(x)$ at $x = 1$ yields

$$f(1) = \frac{1}{\sqrt{1+a}} + \frac{1}{2\sqrt{a(1)^2+1}} = \frac{1}{\sqrt{a+1}} + \frac{1}{2\sqrt{a+1}} = \frac{3}{2\sqrt{a+1}}$$

Then, it remains to see that $f(x)$ attains its maximum value at $x = 1$. We prove it in two ways:

1. Computing the derivative of $f(x)$ with respect to x yields

$$f'(x) = -\frac{1}{2(x+a)^{3/2}} + \frac{1}{2(ax^2+1)^{3/2}}$$

Setting $f'(x) = 0$ to find critical points, we have

$$\frac{1}{(x+a)^{3/2}} = \frac{1}{(ax^2+1)^{3/2}} \Rightarrow x + a = ax^2 + 1$$

Rearranging into a quadratic equation yields

$$ax^2 - x + 1 - a = 0 \Rightarrow (x-1)(ax+a-1) = 0$$

Since $a \geq 1$ and $x > 0$, then the only valid root is $x = 1$. Moreover,

- For $0 < x < 1$, $f'(x) > 0$, so $f(x)$ is increasing.
- For $x > 1$, $f'(x) < 0$, so $f(x)$ is decreasing.

Thus, $f(x)$ attains its global maximum at $x = 1$, and we have

$$f(x) \leq f(1) = \frac{3}{2\sqrt{a+1}}$$

2. Alternatively, since we have

$$f''(x) = \frac{3}{4(x+a)^{5/2}} - \frac{3ax}{2(ax^2+1)^{3/2}} + \frac{3a^2x^3}{2(ax^2+1)^{5/2}}$$

and

$$f''(1) = \frac{3-6a}{4(a+1)^{5/2}} < 0$$

for $a \geq 1$, then $f(x)$ attains its global maximum at $x = 1$.