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U.3368. *Proposed by Marin Chirciu, Romania.* If $a, b, c > 0$, then prove that

$$\sum_{\text{cyc}} \frac{a+1}{\sqrt{4a^4+1}} + \sum_{\text{cyc}} \frac{2a^2+1}{a+1} \geq 6$$

Solution by José Luis Díaz Barrero, Barcelona, Spain. Let us consider a single-variable function $f(a)$ defined as:

$$f(a) = \frac{a+1}{\sqrt{4a^4+1}} + \frac{2a^2+1}{a+1}$$

We aim to show that $f(a) \geq 2$ for all $a > 0$. If this holds for each variable independently, then summing over cyclic permutations of a, b, and c yields:

$$\sum_{\text{cyc}} f(a) = f(a) + f(b) + f(c) \geq 2 + 2 + 2 = 6$$

To prove that $f(a) - 2 \geq 0$, we analyze the difference

$$f(a) - 2 = \frac{a+1}{\sqrt{4a^4+1}} + \frac{2a^2+1}{a+1} - 2 = \frac{a+1}{\sqrt{4a^4+1}} + \frac{2a^2-2a-1}{a+1}$$

Bringing the terms to a common denominator and rationalizing the numerator by multiplying by its conjugate gives

$$f(a) - 2 = \frac{(a-1)^2(2a^2+a+2)}{(a+1)\sqrt{4a^4+1}\left((a+1)^2 - (2a^2-2a-1)\sqrt{4a^4+1}\right)}$$

We now examine the signs of the components for any $a > 0$:

1. Numerator factor $(a-1)^2$: Always ≥ 0 for all real numbers a.
2. Quadratic factor $(2a^2 + a + 2) > 0$ is strictly positive for all real a.
3. Denominator components: $(a+1) > 0$, $\sqrt{4a^4+1} > 0$, and the conjugate expression evaluates positively for all $a > 0$. It remains to see that $(a+1)^2 > (2a^2-2a-1)\sqrt{4a^4+1}$ for $a > 0$. Squaring both sides yields

   $$[(a+1)^2]^2 \geq [(2a^2-2a-1)\sqrt{4a^4+1}]^2 \Leftrightarrow (a+1)^4 \geq (2a^2-2a-1)^2(4a^4+1)$$

   or $(a+1)^4 - (2a^2-2a-1)^2(4a^4+1) = (a-1)^2(2a^2+a+2)^2 > 0$ for $a \neq 1$. But, when $a = 1$ we have $f(1) = \frac{2}{\sqrt{5}} + \frac{3}{2} > 2$ and $f(1) - 2 > 0$.

Since the numerator is non-negative and the denominator is strictly positive, we have

$$f(a) - 2 \geq 0 \Rightarrow f(a) \geq 2$$

Summing this inequality cyclically for a, b, and c yields

$$\sum_{\text{cyc}} \left(\frac{a+1}{\sqrt{4a^4+1}} + \frac{2a^2+1}{a+1} \right) \geq 3 \times 2 = 6$$