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U.3336. *Proposed by Marin Chirciu, Romania.* Solve the following equation in real numbers

$$4^{x^2} + 4^{1+\frac{1}{\sqrt{x}}} = 20$$

Solution by José Luis Díaz Barrero, Barcelona, Spain. First, we determine the domain od possible solutions. Since $\sqrt{x}$ appears in the denominator and inside the square root, the domain is restricted to $x > 0$. Next, we rewrite the second term of the equation as

$$4^{x^2} + 4 \cdot 4^{\frac{1}{\sqrt{x}}} = 20$$

By Inspection, we find that for $x = 1$, we have

$$4^{1^2} + 4^{1+\frac{1}{\sqrt{1}}} = 4^1 + 4^{1+1} = 4 + 4^2 = 4 + 16 = 20$$

Since both sides match, $x = 1$ is a valid solution.

We claim that $x = 1$ is the unique solution. Indeed, we will show that the function

$$f(x) = 4^{x^2} + 4^{1+\frac{1}{\sqrt{x}}} - 20$$

is strictly increasing for $x > 1$. To do that, we examine its first derivative

$$f'(x) = 2x\ln(4)4^{x^2} - \frac{2\ln(4) \cdot 4^{1+\frac{1}{\sqrt{x}}}}{x^{3/2}}$$

and analyzes it for $x > 1$:

1. First Term (Growth): The term $2x\ln(4)4^{x^2}$ grows exponentially as x increases because x^2 increases faster than linearly, making this positive term dominate quickly.

2. Second Term (Decay): The term $\frac{2\ln(4)\cdot 4^{1+\frac{1}{\sqrt{x}}}}{x^{3/2}}$ decreases as x grows, since $\frac{1}{\sqrt{x}}$ shrinks toward 0 and the denominator $x^{3/2}$ increases.

Since the positive growth of the first term outweighs the second term for all $x > 1$, we have

$$f'(x) > 0 \quad \text{for all } x > 1$$

Therefore, the function $f(x)$ is strictly increasing on the interval $(1, \infty)$. This allows us to conclude that the unique real solution of the given equation is $x = 1$.