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U.3328. *Proposed by Vasile Mircea Popa, Romania.* Find

$$\Omega = \lim_{n\to\infty}\left[n\int_0^1 \ln\left(1+e^{-nx}\right)dx\right]$$

Solution by José Luis Díaz Barrero, Barcelona, Spain. Let $u = nx$, which implies $dx = \dfrac{du}{n}$. When $x = 0$, $u = 0$, and when $x = 1$, $u = n$. Then

$$n\int_0^1 \ln\left(1+e^{-nx}\right)dx = \int_0^n \ln\left(1+e^{-u}\right)du$$

Taking the limit turns the definite integral into an improper integral

$$\Omega = \int_0^\infty \ln\left(1+e^{-u}\right)du$$

We compute the integral using integration by parts. Let $v = \ln\left(1+e^{-u}\right)$ and $dw = du$. Then $dv = -\dfrac{1}{e^u+1}\,du$ and $w = u$. Using integration by parts yields

$$\int_0^\infty \ln\left(1+e^{-u}\right)du = \left[u\ln\left(1+e^{-u}\right)\right]_0^\infty + \int_0^\infty \frac{u}{e^u+1}\,du$$

The boundary term evaluates to 0, leaving

$$\Omega = \int_0^\infty \frac{u}{e^u+1}\,du$$

Using the geometric series expansion $\dfrac{1}{e^u+1} = \displaystyle\sum_{k=1}^\infty (-1)^{k-1}e^{-ku}$ we have

$$\Omega = \sum_{k=1}^\infty (-1)^{k-1}\int_0^\infty ue^{-ku}du = \sum_{k=1}^\infty \frac{(-1)^{k-1}}{k^2}$$

This alternating series corresponds to the Dirichlet eta function $\eta(2) = \dfrac{1}{2}\zeta(2)$. Since $\zeta(2) = \dfrac{\pi^2}{6}$ (from the Basel problem), then

$$\Omega = \frac{1}{2}\left(\frac{\pi^2}{6}\right) = \frac{\pi^2}{12}$$