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U.3303. *Proposed by Mihály Bencze and Necualai Stanciu, Romania.* Determine all matrices $A \in M_2(\mathbb{R})$ such that

$$A^3 = \begin{pmatrix} 19 & 30 \\ -45 & -71 \end{pmatrix}$$

Solution by José Luis Díaz Barrero, Barcelona, Spain. To find all matrices $A \in M_2(\mathbb{R})$ such that

$$A^3 = B = \begin{pmatrix} 19 & 30 \\ -45 & -71 \end{pmatrix}$$

we can use the properties of the characteristic polynomial and the Cayley-Hamilton theorem.

1. First, calculate the trace $\operatorname{tr}(B) = -52$) and determinant $(\det(B) = 1$. The characteristic polynomial of B gives the relation between the eigenvalues of A (denoted as λ_1, λ_2) by

$$\lambda_1^3 + \lambda_2^3 = \operatorname{tr}(B) = -52$$
$$\lambda_1^3\lambda_2^3 = \det(B) = 1 \Rightarrow \lambda_1\lambda_2 = 1$$

2. Let $S = \lambda_1 + \lambda_2 = \operatorname{tr}(A)$ and $P = \lambda_1\lambda_2 = \det(A) = 1$. Using the sum of cubes identity, we have

$$\lambda_1^3 + \lambda_2^3 = (\lambda_1 + \lambda_2)^3 - 3\lambda_1\lambda_2(\lambda_1 + \lambda_2)$$

Substituting our values yields

$$-52 = S^3 - 3(1)(S) \Rightarrow S^3 - 3S + 52 = 0 \Leftrightarrow (S + 4)(S^2 - 4S + 13) = 0$$

Since the quadratic factor $S^2 - 4S + 13$ has a negative discriminant ($\Delta = -36$), the only real root is

$$\operatorname{tr}(A) = S = -4 \quad \text{and} \quad \det(A) = 1$$

3. By the Cayley-Hamilton theorem, A satisfies its characteristic equation

$$A^2 - \operatorname{tr}(A)A + \det(A)I = 0 \Rightarrow A^2 + 4A + I = 0$$

Multiplying by A and substituting A^2 we obtain

$$A^3 = -4A^2 - A = 15A + 4I$$

Since $A^3 = B$, we have $B = 15A + 4I$, which gives

$$A = \frac{1}{15}(B - 4I)$$

4. Compute the Final Matrix A. We have

$$A = \frac{1}{15}\left[\begin{pmatrix} 19 & 30 \\ -45 & -71 \end{pmatrix} - \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}\right] = \frac{1}{15}\begin{pmatrix} 15 & 30 \\ -45 & -75 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -3 & -5 \end{pmatrix}$$

Therefore, the unique real matrix $A \in M_2(\mathbb{R})$ satisfying the equation is

$$A = \begin{pmatrix} 1 & 2 \\ -3 & -5 \end{pmatrix}$$