Telescoping Inverse-Sine Sums

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Abstract

We study a class of finite and infinite sums involving the principal inverse-sine function. Beginning with a problem proposed by Mais Hasanov in the *Romanian Mathematical Magazine*, we give a branch-complete telescoping proof and formulate a general identity for adjacent terms of an arbitrary real sequence. A positive radical specialization yields exact finite sums and infinite series with explicitly computable remainders. Further consequences include arithmetic, geometric, hyperbolic, Fibonacci, and rationally parametrized families, together with a method for constructing an inverse-sine series for any prescribed acute angle.

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1 Introduction

Telescoping is one of the most effective elementary mechanisms for the exact summation of a series: the summand is represented as a difference of consecutive quantities, so all intermediate contributions cancel [1, 5]. For sums involving inverse trigonometric functions, however, a formal use of an angle-addition formula is not sufficient. The range of the principal inverse function must also be checked; see, for example, the principal-branch conventions in [2].

The motivating identity below was proposed by Mais Hasanov, and a solution by Amin Hajiyev was circulated by the *Romanian Mathematical Magazine* [4]. Our purposes are to provide a fully justified proof, isolate the underlying general theorem, and derive new families of exact identities. Such formulas belong naturally to the classical study of finite and infinite sums represented in standard tables and monographs such as [3].

2 The problem (Mais Hasanov, Romanian Mathematical Magazine)

The problem asks for a proof of the following identity.

For every positive integer k, prove that

$$\arcsin\left(\frac{\sqrt{3}}{2}\right)+\arcsin\left(\frac{\sqrt{8}-\sqrt{3}}{6}\right)+\cdots+\arcsin\left(\frac{\sqrt{k^2+2k}-\sqrt{k^2-1}}{k(k+1)}\right)=\arccos\left(\frac{1}{k+1}\right). \quad (1)$$

Here and throughout, arcsin and arccos denote their principal values.

The main idea is that each summand is the difference of two consecutive angles. Before using this idea, however, one must deal carefully with the principal branch of arcsin.

3 The inverse-sine difference lemma

Lemma 1. *If $0 \le y \le x \le 1$, then*

$$\arcsin\left(x\sqrt{1-y^2}-y\sqrt{1-x^2}\right)=\arcsin x-\arcsin y. \quad (2)$$

Proof. Put $\alpha=\arcsin x$ and $\beta=\arcsin y$. Since $0\le y\le x\le 1$, we have

$$0\le\beta\le\alpha\le\frac{\pi}{2}, \qquad 0\le\alpha-\beta\le\frac{\pi}{2}.$$

In particular, $\alpha-\beta$ lies in the interval on which arcsin is the inverse of sin. Moreover,

$$\cos\alpha=\sqrt{1-x^2}, \qquad \cos\beta=\sqrt{1-y^2}.$$

The sine subtraction formula therefore gives

$$\sin(\alpha-\beta)=x\sqrt{1-y^2}-y\sqrt{1-x^2}.$$

Applying the principal arcsin to both sides yields (2). □

Remark. The hypotheses in Lemma 1 are essential. The formula $\arcsin(\sin t)=t$ is not true for arbitrary t; it is true when $t\in[-\pi/2,\pi/2]$. Thus the branch check above is a necessary part of a complete proof.

4 Detailed solution of the proposed identity

Write the nth argument on the left-hand side of (1) as

$$u_n=\frac{\sqrt{(n+1)^2-1}-\sqrt{n^2-1}}{n(n+1)}, \qquad 1\le n\le k.$$

This notation gives

$$u_1=\frac{\sqrt{3}}{2}, \qquad u_2=\frac{\sqrt{8}-\sqrt{3}}{6},$$

and

$$u_k=\frac{\sqrt{k^2+2k}-\sqrt{k^2-1}}{k(k+1)},$$

so the sum in the problem is exactly

$$S_k = \sum_{n=1}^{k} \arcsin u_n.$$

Define

$$\alpha_n = \arcsin\left(\frac{1}{n}\right), \qquad n \geq 1.$$

The sequence (α_n) is decreasing and lies in $(0, \pi/2]$. Also,

$$\sin \alpha_n = \frac{1}{n}, \qquad \cos \alpha_n = \sqrt{1 - \frac{1}{n^2}} = \frac{\sqrt{n^2 - 1}}{n}.$$

Consequently,

$$\begin{aligned}
\sin(\alpha_n - \alpha_{n+1}) &= \sin \alpha_n \cos \alpha_{n+1} - \cos \alpha_n \sin \alpha_{n+1} \\
&= \frac{1}{n} \frac{\sqrt{(n+1)^2 - 1}}{n+1} - \frac{\sqrt{n^2 - 1}}{n} \frac{1}{n+1} \\
&= \frac{\sqrt{(n+1)^2 - 1} - \sqrt{n^2 - 1}}{n(n+1)} \\
&= u_n.
\end{aligned}$$

Because $0 \leq \alpha_n - \alpha_{n+1} \leq \pi/2$, taking the principal inverse sine introduces no ambiguity. Hence

$$\arcsin u_n = \alpha_n - \alpha_{n+1}.$$

The sum now telescopes:

$$\begin{aligned}
S_k &= \sum_{n=1}^{k} (\alpha_n - \alpha_{n+1}) \\
&= \alpha_1 - \alpha_{k+1} \\
&= \frac{\pi}{2} - \arcsin\left(\frac{1}{k+1}\right) \\
&= \arccos\left(\frac{1}{k+1}\right).
\end{aligned}$$

This proves (1).

5 General telescoping theorems

The mechanism behind the proposed problem does not require positivity or monotonicity. Its most general elementary form is expressed directly in terms of adjacent values in $[-1, 1]$.

Theorem 1 (Signed sequence form)**.** *Let $x_1, x_2, \ldots, x_{N+1} \in [-1, 1]$, and suppose that*

$$|\arcsin x_j - \arcsin x_{j+1}| \leq \frac{\pi}{2}, \qquad 1 \leq j \leq N. \tag{3}$$

Then

$$\sum_{j=1}^{N} \arcsin\Big(x_j\sqrt{1-x_{j+1}^2} - x_{j+1}\sqrt{1-x_j^2}\Big)$$
$$= \arcsin x_1 - \arcsin x_{N+1}. \tag{4}$$

No monotonicity of the sequence (x_j) is required.

Proof. Let $\theta_j = \arcsin x_j \in [-\pi/2, \pi/2]$. On this interval $\cos\theta_j = \sqrt{1-x_j^2}$, and therefore

$$\begin{aligned} x_j\sqrt{1-x_{j+1}^2} - x_{j+1}\sqrt{1-x_j^2} &= \sin\theta_j\cos\theta_{j+1} - \cos\theta_j\sin\theta_{j+1} \\ &= \sin(\theta_j - \theta_{j+1}). \end{aligned}$$

Condition (3) places $\theta_j - \theta_{j+1}$ in the principal range of arcsin. Hence the jth summand is $\theta_j - \theta_{j+1}$. Summing gives

$$(\theta_1 - \theta_2) + (\theta_2 - \theta_3) + \cdots + (\theta_N - \theta_{N+1}) = \theta_1 - \theta_{N+1},$$

which is (4). $\square$

For the radical form relevant to the original problem, the integers $1, 2, \ldots, k+1$ can be replaced by any nondecreasing positive sequence.

Theorem 2 (Positive radical form)**.** *Let $c > 0$, and let*

$$c \le a_1 \le a_2 \le \cdots \le a_{N+1}.$$

Then

$$\sum_{j=1}^{N} \arcsin\left(\frac{c\big(\sqrt{a_{j+1}^2 - c^2} - \sqrt{a_j^2 - c^2}\big)}{a_j a_{j+1}}\right)$$
$$= \arcsin\left(\frac{c}{a_1}\right) - \arcsin\left(\frac{c}{a_{N+1}}\right). \tag{5}$$

If $a_1 = c$, this simplifies to

$$\sum_{j=1}^{N} \arcsin\left(\frac{c\big(\sqrt{a_{j+1}^2 - c^2} - \sqrt{a_j^2 - c^2}\big)}{a_j a_{j+1}}\right) = \arccos\left(\frac{c}{a_{N+1}}\right). \tag{6}$$

Proof. Set

$$\theta_j = \arcsin\left(\frac{c}{a_j}\right).$$

Since (a_j) is nondecreasing, (θ_j) is nonincreasing, and $0 \le \theta_j - \theta_{j+1} \le \pi/2$. Furthermore,

$$\sin(\theta_j - \theta_{j+1}) = \frac{c}{a_j}\frac{\sqrt{a_{j+1}^2 - c^2}}{a_{j+1}} - \frac{\sqrt{a_j^2 - c^2}}{a_j}\frac{c}{a_{j+1}}$$

$$= \frac{c\left(\sqrt{a_{j+1}^2 - c^2} - \sqrt{a_j^2 - c^2}\right)}{a_j a_{j+1}}.$$

The branch condition allows us to apply arcsin, so the jth summand in (5) is $\theta_j - \theta_{j+1}$. Telescoping gives $\theta_1 - \theta_{N+1}$, which proves (5). When $a_1 = c$, $\theta_1 = \pi/2$, and the complementary-angle identity gives (6). □

An immediately useful specialization is an arithmetic progression.

Corollary 1 (Arithmetic-progression form)**.** *Let $c > 0$, $A \geq c$, $d > 0$, and $N \geq 1$. Then*

$$\sum_{j=0}^{N-1} \arcsin\left(\frac{c\left(\sqrt{(A+(j+1)d)^2 - c^2} - \sqrt{(A+jd)^2 - c^2}\right)}{(A+jd)(A+(j+1)d)}\right)$$
$$= \arcsin\left(\frac{c}{A}\right) - \arcsin\left(\frac{c}{A+Nd}\right).$$

The original problem is the case $A = c = d = 1$ and $N = k$.

The following parametrization produces rational arguments whenever the parameters are rational.

Corollary 2 (Rational parametrization)**.** *Let*

$$1 \leq r_1 \leq r_2 \leq \cdots \leq r_{N+1}.$$

Then

$$\sum_{j=1}^{N} \arcsin\left(\frac{2(r_{j+1} - r_j)(1 + r_j r_{j+1})}{(1 + r_j^2)(1 + r_{j+1}^2)}\right)$$
$$= 2\arctan\left(\frac{1}{r_1}\right) - 2\arctan\left(\frac{1}{r_{N+1}}\right). \tag{7}$$

In particular, if all the r_j are rational, every argument of arcsin *in* (7) *is rational.*

Proof. In Theorem 2, set

$$a_j = \frac{c(r_j^2 + 1)}{2r_j}.$$

For $r_j \geq 1$, the sequence (a_j) is nondecreasing, and

$$\sqrt{a_j^2 - c^2} = \frac{c(r_j^2 - 1)}{2r_j}, \qquad \arcsin\left(\frac{c}{a_j}\right) = 2\arctan\left(\frac{1}{r_j}\right).$$

Substitution and simplification give (7). □

6 Applications and new identities

6.1 General convergence and rate control

The freedom in choosing (a_j) determines not only the summands but also the rate of convergence.

Proposition 1 (Exact remainder)**.** *Under the assumptions of Theorem 2, suppose additionally that $a_j \to \infty$. Then*

$$\sum_{j=1}^{\infty} \arcsin\left(\frac{c\big(\sqrt{a_{j+1}^2 - c^2} - \sqrt{a_j^2 - c^2}\big)}{a_j a_{j+1}}\right) = \arcsin\left(\frac{c}{a_1}\right). \tag{8}$$

The remainder after N terms is

$$R_N = \arcsin\left(\frac{c}{a_{N+1}}\right) \sim \frac{c}{a_{N+1}}. \tag{9}$$

Whenever $a_{N+1} > c$, it satisfies the explicit bounds

$$\frac{c}{a_{N+1}} \le R_N \le \frac{c}{\sqrt{a_{N+1}^2 - c^2}}. \tag{10}$$

Proof. Let $N \to \infty$ in (5). Since $c/a_{N+1} \to 0$, the second endpoint tends to zero, proving (8). Formula (9) follows by subtracting the Nth partial sum. Finally, for $0 \le x < 1$,

$$x \le \arcsin x \le \frac{x}{\sqrt{1-x^2}},$$

and substitution of $x = c/a_{N+1}$ gives (10). The asymptotic relation follows from $\arcsin x \sim x$ as $x \to 0$. □

Thus $a_n \sim Cn^p$ gives an error asymptotic to $(c/C)n^{-p}$, whereas $a_n \sim Cq^n$ with $q > 1$ gives an error asymptotic to $(c/C)q^{-n}$. The theorem therefore allows one to design the convergence rate in advance.

6.2 An infinite series and an exact error estimate

Letting $k \to \infty$ in (1) gives

$$\sum_{n=1}^{\infty} \arcsin\left(\frac{\sqrt{(n+1)^2 - 1} - \sqrt{n^2 - 1}}{n(n+1)}\right) = \frac{\pi}{2}. \tag{11}$$

After the first N terms, the remainder is known exactly:

$$R_N = \frac{\pi}{2} - S_N = \arcsin\left(\frac{1}{N+1}\right).$$

For $0 \le x < 1$,

$$x \le \arcsin x \le \frac{x}{\sqrt{1-x^2}}.$$

Indeed, this follows by bounding the integrand in $\arcsin x = \int_0^x (1-t^2)^{-1/2}\,dt$. Thus, for $N \geq 1$,

$$\frac{1}{N+1} \leq R_N \leq \frac{1}{\sqrt{(N+1)^2-1}}. \tag{12}$$

This gives both convergence and explicit control of the truncation error.

6.3 A family with rational arguments

The radical arguments in the original problem can be replaced by rational ones. In Corollary 2, take $r_j = j$; equivalently, in Theorem 2, take $c = 1$ and

$$a_j = \frac{j^2+1}{2j}, \qquad j \geq 1.$$

This sequence begins at $a_1 = 1$, is increasing, and satisfies

$$\sqrt{a_j^2 - 1} = \frac{j^2-1}{2j}.$$

Substitution into (6), followed by simplification, gives the entirely rational identity

$$\boxed{\sum_{j=1}^{N} \arcsin\left(\frac{2(j^2+j+1)}{(j^2+1)((j+1)^2+1)}\right) = \arccos\left(\frac{2(N+1)}{(N+1)^2+1}\right).} \tag{13}$$

Consequently,

$$\sum_{j=1}^{\infty} \arcsin\left(\frac{2(j^2+j+1)}{(j^2+1)((j+1)^2+1)}\right) = \frac{\pi}{2}.$$

The exact tail after N terms is

$$\arcsin\left(\frac{2(N+1)}{(N+1)^2+1}\right) = 2\arctan\left(\frac{1}{N+1}\right).$$

6.4 A geometric family

Fix $q > 1$ and choose the sequence $1, q, q^2, \ldots, q^N$ in Theorem 2. We obtain

$$\boxed{\sum_{j=0}^{N-1} \arcsin\left(\frac{\sqrt{q^{2j+2}-1} - \sqrt{q^{2j}-1}}{q^{2j+1}}\right) = \arccos(q^{-N}).} \tag{14}$$

Letting $N \to \infty$ again produces an infinite series with sum $\pi/2$. For example, setting $q = 2$ yields

$$\arcsin\left(\frac{\sqrt{3}}{2}\right) + \arcsin\left(\frac{\sqrt{15}-\sqrt{3}}{8}\right) + \arcsin\left(\frac{\sqrt{63}-\sqrt{15}}{32}\right) + \cdots = \frac{\pi}{2}.$$

By Proposition 1, the error is asymptotic to q^{-N}.

6.5 A hyperbolic family

Let

$$0 = t_0 \le t_1 \le \cdots \le t_N$$

and set $a_{j+1} = c \cosh t_j$ in Theorem 2. Since $\sqrt{a_{j+1}^2 - c^2} = c \sinh t_j$, we obtain

$$\boxed{\sum_{j=0}^{N-1} \arcsin\left(\frac{\sinh t_{j+1} - \sinh t_j}{\cosh t_j \cosh t_{j+1}}\right) = \arccos(\operatorname{sech} t_N).} \tag{15}$$

For a uniform mesh $t_j = jh$, $h > 0$, this becomes

$$\sum_{j=0}^{N-1} \arcsin\left(\frac{\sinh((j+1)h) - \sinh(jh)}{\cosh(jh)\cosh((j+1)h)}\right) = \arccos(\operatorname{sech}(Nh)).$$

As $N \to \infty$, the sum tends to $\pi/2$, and its remainder is

$$\arcsin(\operatorname{sech}(Nh)) \sim 2e^{-Nh}.$$

6.6 A Fibonacci family

Let $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$; see, for example, [6]. Taking $a_j = F_{j+1}$ and $c = 1$ in Theorem 2 gives

$$\boxed{\sum_{j=1}^{N} \arcsin\left(\frac{\sqrt{F_{j+2}^2 - 1} - \sqrt{F_{j+1}^2 - 1}}{F_{j+1}F_{j+2}}\right) = \arccos\left(\frac{1}{F_{N+2}}\right).} \tag{16}$$

Consequently,

$$\arcsin\left(\frac{\sqrt{3}}{2}\right) + \arcsin\left(\frac{\sqrt{8} - \sqrt{3}}{6}\right) + \arcsin\left(\frac{\sqrt{24} - \sqrt{8}}{15}\right) + \cdots$$
$$= \frac{\pi}{2}.$$

The exact tail is $\arcsin(1/F_{N+2})$. If $\rho = (1 + \sqrt{5})/2$ is the golden ratio, Binet's formula gives

$$\arcsin\left(\frac{1}{F_{N+2}}\right) \sim \sqrt{5}\,\rho^{-(N+2)},$$

so this family converges geometrically.

6.7 Constructing a series for any acute angle

The general theorem is also a design principle. Let $0 < \varphi \le \pi/2$, choose $c > 0$, and choose any nondecreasing sequence (a_j) such that

$$a_1 = \frac{c}{\sin\varphi}, \qquad a_j \longrightarrow \infty.$$

Passing to the limit in (5) gives

$$\sum_{j=1}^{\infty} \arcsin\left(\frac{c\left(\sqrt{a_{j+1}^2 - c^2} - \sqrt{a_j^2 - c^2}\right)}{a_j a_{j+1}}\right) = \varphi. \tag{17}$$

Thus a suitable choice of the sequence (a_j) creates an exact telescoping inverse-sine expansion for any prescribed acute angle.

7 Conclusion

The expression in the proposed problem is engineered from the sine of a difference of consecutive angles:

$$\arcsin\left(\frac{1}{n}\right) - \arcsin\left(\frac{1}{n+1}\right).$$

Once the principal-branch condition is checked, the identity follows by telescoping. The signed sequence identity, Theorem 1, shows that neither positivity nor monotonicity is intrinsic to the method; only the adjacent branch condition matters. The positive radical form, Theorem 2, then provides a practical generator of exact finite sums and convergent series. Rational, geometric, hyperbolic, and Fibonacci choices illustrate how algebraic structure and convergence rate can be prescribed through the underlying sequence.

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