ROMANIAN MATHEMATICAL MAGAZINE

SP.614 Let be $A(1,3,2); B(3,1,1); C(4,2,0)$. Find the area of the parallelogram builded on $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$\overrightarrow{AB} = (x_B - x_A)\vec{i} + (y_B - y_A)\vec{j} + (z_B - z_A)\vec{k}$$

$$\overrightarrow{AB} = (3-1)\vec{i} + (1-3)\vec{j} + (1-2)\vec{k}$$

$$\overrightarrow{AB} = 2\vec{i} - 2\vec{j} - \vec{k}$$

$$\overrightarrow{AC} = (x_C - x_A)\vec{i} + (y_C - y_A)\vec{j} + (z_C - z_A)\vec{k}$$

$$\overrightarrow{AC} = (4-1)\vec{i} + (2-3)\vec{j} + (1-2)\vec{k}$$

$$\overrightarrow{AC} = 3\vec{i} - \vec{j} - 2\vec{k}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & -1 \\ 3 & -1 & -2 \end{vmatrix} = 4\vec{i} - 3\vec{j} - 2\vec{k} + 6\vec{k} - \vec{i} + 4\vec{j} = 3\vec{i} + \vec{j} + 4\vec{k}$$

Let be $ABCD$ the asked parallelogram.

$$\text{Area}[ABCD] = \left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \sqrt{3^2 + 1^2 + 4^2} = \sqrt{26}$$

Solution 2 by José Luis Díaz – Barrero – Spain

The area of a parallelogram spanned by two vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is equal to the magnitude (norm) of their cross product:

$$Area = \left|\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|\right|$$

Now, we find the components of the vectors

$$\overrightarrow{AB} = (3-1, 1-3, 1-2) = (2, -2, -1)$$

$$\overrightarrow{AC} = (4-1, 2-3, 0-2) = (3, -1, -2)$$

The cross product is given by

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 2 & -2 & -1 \\ 3 & -1 & -2 \end{vmatrix} = (3, 1, 4)$$

The area of the parallelogram S is the magnitude of this vector.

That is $S=\left|\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|\right|=\sqrt{3^2+1^2+4^2}=\sqrt{26}$

Solution 3 by Marin Chirciu-Romania

We use: $S[\overrightarrow{AB},\overrightarrow{AC}]=\left|\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|\right|$

In our case:

$$\overrightarrow{AB}=(3-1)\vec{i}+(1-3)\vec{j}+(1-2)\vec{k}=2\vec{i}-2\vec{j}-\vec{k}\Rightarrow$$

$$\Rightarrow\left|\left|\overrightarrow{AB}\right|\right|=\sqrt{2^2+(-2)^2+(-1)^2}=\sqrt{9}=3$$

$$\overrightarrow{AC}=(4-1)\vec{i}+(2-3)\vec{j}+(0-2)\vec{k}=3\vec{i}-\vec{j}-2\vec{k}\Rightarrow$$

$$\Rightarrow\left|\left|\overrightarrow{AC}\right|\right|=\sqrt{3^2+(-1)^2+(-2)^2}=\sqrt{14}$$

$S=\left|\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|\right|=\left|\left|\overrightarrow{AB}\right|\right|\cdot\left|\left|\overrightarrow{AC}\right|\right|\cdot\sin\angle(\overrightarrow{AB},\overrightarrow{AC})=3\cdot\sqrt{14}\cdot\frac{\sqrt{13}}{3\sqrt{7}}=\sqrt{26}$, see:

$$\cos\angle(\overrightarrow{AB},\overrightarrow{AC})=\frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\left|\left|\overrightarrow{AB}\right|\right|\cdot\left|\left|\overrightarrow{AC}\right|\right|}=\frac{(2\vec{i}-2\vec{j}-\vec{k})\cdot(3\vec{i}-\vec{j}-2\vec{k})}{3\cdot\sqrt{14}}=$$

$$=\frac{2\cdot3+(-2)(-1)+(-1)(-2)}{3\cdot\sqrt{14}}=\frac{10}{3\cdot\sqrt{14}}\Rightarrow\sin\angle(\overrightarrow{AB},\overrightarrow{AC})=\frac{\sqrt{13}}{3\sqrt{7}}$$