ROMANIAN MATHEMATICAL MAGAZINE

SP.612 Let be $A(1,2,3); B(4,1,1); C(0,3,2); D(2,1,3)$. Find the volume of the parallelepiped builded on the vectors $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$.

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$\overrightarrow{AB} = (x_B - x_A)\vec{i} + (y_B - y_A)\vec{j} + (z_B - z_A)\vec{k}$$

$$\overrightarrow{AB} = (4-1)\vec{i} + (1-2)\vec{j} + (1-3)\vec{k} = 3\vec{i} - \vec{j} - 2\vec{k}$$

$$\overrightarrow{AC} = (x_C - x_A)\vec{i} + (y_C - y_A)\vec{j} + (z_C - z_A)\vec{k}$$

$$\overrightarrow{AC} = (0-1)\vec{i} + (3-2)\vec{j} + (2-3)\vec{k} = -\vec{i} + \vec{j} - \vec{k}$$

$$\overrightarrow{AD} = (x_D - x_A)\vec{i} + (y_D - y_A)\vec{j} + (z_D - z_A)\vec{k}$$

$$\overrightarrow{AD} = (2-1)\vec{i} + (1-2)\vec{j} + (3-3)\vec{k} = \vec{i} - \vec{j}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & -2 \\ -1 & 1 & -1 \end{vmatrix} = \vec{i} + 2\vec{j} + 3\vec{k} - \vec{k} + 3\vec{j} + 2\vec{i}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = 3\vec{i} + 5\vec{j} + 2\vec{k}$$

$$\overrightarrow{AD} \cdot \left(\overrightarrow{AB} \times \overrightarrow{AC}\right) = 1 \cdot 3 - 1 \cdot 5 = -2$$

$$V[parallelipiped] = |-2| = 2$$

Solution 2 by Jose Luis Diaz Barrero-Spain

To find the volume V of the parallelepiped built on the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{AD}$ from the points $A(1,2,3)$, $B(4,1,1)$, $C(0,3,2)$, and $D(2,1,3)$, w use the scalar triple product. The volume is given by the absolute value of the determinant of the matrix formed by these three vectors:

$$V = \left|\overrightarrow{AB} \cdot \left(\overrightarrow{AC} \times \overrightarrow{AD}\right)\right|$$

We find each vector by subtracting the coordinates of the initial point from the terminal point:

$$\overrightarrow{AB} = (4-1, 1-2, 1-3) = (3, -1, -2)$$

$$\overrightarrow{AC} = (0-1, 3-2, 2-3) = (-1, 1, -1)$$

$$\overrightarrow{AD} = (2-1, 1-2, 3-3) = (1, -1, 0)$$

The volume is the absolute value of the determinant of the 3×3 matrix containing these vectors as rows:

$$V = \left|\det\begin{pmatrix} 3 & -1 & -2 \\ -1 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}\right| = |-2| = 2$$

Solution 3 by Marin Chirciu-Romania

We use: $V[\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}] = |(\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD})| = |\overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD})|$.

In our case:

$$\overrightarrow{AB} = (4-1)\vec{i} + (1-2)\vec{j} + (1-3)\vec{k} = 3\vec{i} - \vec{j} - 2\vec{k} \Rightarrow$$

$$\Rightarrow \left|\left|\overrightarrow{AB}\right|\right| = \sqrt{3^2 + (-1)^2 + (-2)^2} = \sqrt{14}$$

$$\overrightarrow{AC} = (0-1)\vec{i} + (3-2)\vec{j} + (2-3)\vec{k} = -\vec{i} + \vec{j} - \vec{k} \Rightarrow$$

$$\Rightarrow \left|\left|\overrightarrow{AC}\right|\right| = \sqrt{(-1)^2 + 1^2 + (-1)^2} = \sqrt{3}.$$

$$\overrightarrow{AD} = (2-1)\vec{i} + (1-2)\vec{j} + (3-3)\vec{k} = \vec{i} - \vec{j} \Rightarrow$$

$$\Rightarrow \left|\left|\overrightarrow{AD}\right|\right| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & -2 \\ -1 & 1 & -1 \end{vmatrix} = \vec{i} + 2\vec{j} + 3\vec{k} - \vec{k} + 3\vec{j} + 2\vec{i}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = 3\vec{i} + 5\vec{j} + 2\vec{k}$$

$$\overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{AC}) = 1 \cdot 3 - 1 \cdot 5 = -2$$

$$V[parallelipiped] = |-2| = 2$$