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SP.611. *Proposed by Daniel Sitaru, Romania.* If $x, y, z > 0$ then

$$(x^5 + y^5 + z^5)(x^6 + y^6 + z^6)(x^2 + y^2 + z^2)^5 \geq (x^3 + y^3 + z^3)^7.$$

Solution by José Luis Díaz-Barrero, Barcelona, Spain. Hölder's Inequality for multiple sequences states that for positive numbers and weights that sum to 1, the following holds:

$$\prod_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ji} \right)^{w_j} \geq \sum_{i=1}^{n} \prod_{j=1}^{m} a_{ji}^{w_j} \quad \text{where } \sum_{j=1}^{m} w_j = 1$$

We can interpret the LHS as a product of 7 brackets in total, assigning each an equal weight of $w_j = \frac{1}{7}$. That is,

- 1 bracket of $(x^5 + y^5 + z^5)$
- 1 bracket of $(x^6 + y^6 + z^6)$
- 5 brackets of $(x^2 + y^2 + z^2)$

Applying Hölder's Inequality with these terms yields:

$$(x^5 + y^5 + z^5)^{\frac{1}{7}} \cdot (x^6 + y^6 + z^6)^{\frac{1}{7}} \cdot (x^2 + y^2 + z^2)^{\frac{5}{7}} \geq x^P + y^P + z^P,$$

where the exponent P for each variable is calculated by multiplying its original powers by their respective weights:

$$P = \left(5 \times \frac{1}{7}\right) + \left(6 \times \frac{1}{7}\right) + \left(2 \times \frac{5}{7}\right) = \frac{5 + 6 + 10}{7} = \frac{21}{7} = 3$$

Substituting $P = 3$ back into the inequality gives:

$$(x^5 + y^5 + z^5)^{\frac{1}{7}} (x^6 + y^6 + z^6)^{\frac{1}{7}} (x^2 + y^2 + z^2)^{\frac{5}{7}} \geq x^3 + y^3 + z^3$$

Since all terms are positive, we can raise both sides to the 7^{th} power to obtain the final result:

$$(x^5 + y^5 + z^5)(x^6 + y^6 + z^6)(x^2 + y^2 + z^2)^5 \geq (x^3 + y^3 + z^3)^7$$

Equality holds if and only if $x = y = z$.