ROMANIAN MATHEMATICAL MAGAZINE

SP.611 If $x, y, z > 0$ then:

$$\left(x^5+y^5+z^5\right)\left(x^6+y^6+z^6\right)\left(x^2+y^2+z^2\right)^5 \geq \left(x^3+y^3+z^3\right)^7$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

Let be $f:(0,\infty)\to\mathbb{R}; f(x)=x^3,\ f'(x)=3x^2; f''(x)=6x>0\Rightarrow f$ convex

By Jensen's inequality: $p_1f(x_1)+p_2f(x_2)+p_3f(x_3)\geq f(p_1x_1+p_2x_2+p_3x_3)$ (1)

$$\text{for } p_1,p_2,p_3>0; p_1+p_2+p_3=1$$

Replace in (1):

$$p_1=\frac{x^2}{x^2+y^2+z^2}; p_2=\frac{y^2}{x^2+y^2+z^2}; p_3=\frac{z^2}{x^2+y^2+z^2}$$

$$\sum_{cyc}\frac{x^2}{x^2+y^2+z^2}\cdot x^3\geq f\left(\sum_{cyc}\frac{x^2}{x^2+y^2+z^2}\cdot x\right)$$

$$\frac{1}{x^2+y^2+z^2}\cdot\sum_{cyc}x^5\geq\left(\sum_{cyc}\frac{x^3}{x^2+y^2+z^2}\right)^3$$

$$\frac{x^5+y^5+z^5}{x^2+y^2+z^2}\geq\frac{(x^3+y^3+z^3)^3}{(x^2+y^2+z^2)^3}$$

$$\left(x^5+y^5+z^5\right)(x^2+y^2+z^2)^2\geq(x^3+y^3+z^3)^3 \quad (2)$$

Let be $g:(0,\infty)\to\mathbb{R}; g(x)=x^4$

$g'(x)=4x^3; g''(x)=12x^2>0\Rightarrow g$ convex

By Jensen's inequality:

$$q_1g(x_1)+q_2g(x_2)+q_3g(x_3)\geq g(q_1x_1+q_2x_2+q_3x_3)$$

Replace in (3):

$$q_1=\frac{x^2}{x^2+y^2+z^2}; q_2=\frac{y^2}{x^2+y^2+z^2}; q_3=\frac{x^2}{x^2+y^2+z^2}$$

$$\sum_{cyc}\frac{x^2}{x^2+y^2+z^2}\cdot x^4\geq g\left(\sum_{cyc}\frac{x^2}{x^2+y^2+z^2}\cdot x\right)$$

$$\frac{x^6+y^6+z^6}{x^2+y^2+z^2}\geq\frac{(x^3+y^3+z^3)^4}{(x^2+y^2+z^2)^4}$$

$$(x^6+y^6+z^6)(x^2+y^2+z^2)^3\geq(x^3+y^3+z^3)^4 \quad (4)$$

By multiplying (2); (4): $(x^5+y^5+z^5)(x^6+y^6+z^6)(x^2+y^2+z^2)^5 \geq (x^3+y^3+z^3)^7$

Equality holds for $x=y=z$.

Solution 2 by Jose Luis Diaz – Barrero – Spain

Holder's Inequality for multiple sequences states that for positive numbers and weights that sum to 1 the following holds:

$$\prod_{j=1}^{m}\left(\sum_{i}^{n} a_{ji}\right)^{w_j} \geq \sum_{i=1}^{n}\prod_{j=1}^{m} a_{ij}^{w_j} \quad \text{where } \sum_{j=1}^{m} w_j = 1$$

We can interpret the LHS as a product of 7 brackets in total, assigning each an equal weight of $w_j = \frac{1}{7}$. **That is**

- **1 bracket of** $(x^5+y^5+z^5)$
- **1 bracket of** $(x^6+y^6+z^6)$
- **5 brackets of** $(x^2+y^2+z^2)$

Applying Holder's Inequality with these terms yields:

$$(x^5+y^5+z^5)^{\frac{1}{7}} \cdot (x^6+y^6+z^6)^{\frac{1}{7}} \cdot (x^2+y^2+z^2)^{\frac{5}{7}} \geq x^P + y^P + z^P$$

where the exponent P **for each variable is calculated by multiplying its original powers by their respective weights:**

$$P = \left(5 \times \frac{1}{7}\right) + \left(6 \times \frac{1}{7}\right) + \left(2 \times \frac{5}{7}\right) = \frac{5+6+10}{7} = \frac{21}{7} = 3$$

Substituting $P = 3$ **back into the inequality gives:**

$$(x^5+y^5+z^5)^{\frac{1}{7}}(x^6+y^6+z^6)^{\frac{1}{7}}(x^2+y^2+z^2)^{\frac{5}{7}} \geq x^3+y^3+z^3$$

Since all terms are positive, we can raise both sides to the 7^{th} **power to obtain the final result:**

$$(x^5+y^5+z^5)(x^6+y^6+z^6)(x^2+y^2+z^2)^5 \geq (x^3+y^3+z^3)^7$$

Equality holds if and only if $x=y=z$.

Solution 3 by Marin Chirciu-Romania

$$LHS = \sum x^5 \sum x^6 \left(\sum x^2\right)^5 \overset{Holder}{\geq} \left(\sum \sqrt[7]{x^{5+6+2\cdot 5}}\right)^7 = \left(\sum x^3\right)^7 = RHS\,.$$

Equality holds for $x=y=z$.