ROMANIAN MATHEMATICAL MAGAZINE

SP.610 If $a, b, c > 0$ then:

$$\sqrt{a^2+1}+\sqrt{b^2+1}+\sqrt{c^2+1} \geq \sqrt{(a+b+c)^2+9}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

Let be $f: (0, \infty) \to \mathbb{R}; f(x) = \sqrt{x^2+1}$

$f'(x) = \frac{x}{\sqrt{x^2+1}}; f''(x) = \frac{1}{(x^2+1)\sqrt{x^2+1}} > 0; f$ convex

By Jensen's inequality:

$$\frac{\sum_{cyc} af\left(\frac{1}{a}\right)}{\sum_{cyc} a} \geq f\left(\frac{\sum_{cyc} a \cdot \frac{1}{a}}{\sum_{cyc} a}\right) = f\left(\frac{3}{\sum_{cyc} a}\right)$$

$$\sum_{cyc} af\left(\frac{1}{a}\right) \geq (a+b+c)f\left(\frac{3}{a+b+c}\right)$$

$$af\left(\frac{1}{a}\right)+bf\left(\frac{1}{b}\right)+cf\left(\frac{1}{c}\right) \geq (a+b+c)f\left(\frac{3}{a+b+c}\right)$$

$$a\sqrt{\frac{1}{a^2}+1}+b\sqrt{\frac{1}{b^2}+1}+c\sqrt{\frac{1}{c^2}+1} \geq (a+b+c)\sqrt{\left(\frac{3}{a+b+c}\right)^2+1}$$

$$\sqrt{1+a^2}+\sqrt{1+b^2}+\sqrt{1+c^2} \geq \sqrt{9+(a+b+c)^2}$$

Equality holds for $a = b = c = 1$.

Solution 2 by Jose Luis Diaz – Barrero – Spain

Minkowski's Inequality states that for any real numbers:

$$\sqrt{x_1^2+y_1^2}+\sqrt{x_2^2+y_2^2}+\sqrt{x_3^2+y_3^2} \geq \sqrt{(x_1+x_2+x_3)^2+(y_1+y_2+y_3)^2}$$

Choosing the components (x_i, y_i) to match the terms inside each radical on the left-hand side:

- For $\sqrt{a^2+1}$, let $x_1 = a$ and $y_1 = 1$.
- For $\sqrt{b^2+1}$, let $x_2 = b$ and $y_2 = 1$.
- For $\sqrt{c^2+1}$, let $x_3 = c$ and $y_3 = 1$.

Substituting these components into Minkowski's Inequality gives:

$$\sqrt{a^2+1}+\sqrt{b^2+1}+\sqrt{c^2+1}\geq\sqrt{(a+b+c)^2+(1+1+1)^2}=\sqrt{(a+b+c)^2+9}$$

Equality holds if and only if the vectors $(a,1),(b,1)$**, and** $(c,1)$ **are proportional:**

$$\frac{a}{1}=\frac{b}{1}=\frac{c}{1}\Rightarrow a=b=c$$

Thus, equality holds if and only if $a=b=c$**.**