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SP.610. *Proposed by Daniel Sitaru, Romania.* If $a, b, c > 0$ then

$$\sqrt{a^2+1}+\sqrt{b^2+1}+\sqrt{c^2+1} \geq \sqrt{(a+b+c)^2+9}.$$

Solution by José Luis Díaz-Barrero, Barcelona, Spain. Minkowski's Inequality states that for any real numbers:

$$\sqrt{x_1^2+y_1^2}+\sqrt{x_2^2+y_2^2}+\sqrt{x_3^2+y_3^2} \geq \sqrt{(x_1+x_2+x_3)^2+(y_1+y_2+y_3)^2}$$

Choosing the components (x_i, y_i) to match the terms inside each radical on the left-hand side:

- For $\sqrt{a^2+1}$, let $x_1 = a$ and $y_1 = 1$.
- For $\sqrt{b^2+1}$, let $x_2 = b$ and $y_2 = 1$.
- For $\sqrt{c^2+1}$, let $x_3 = c$ and $y_3 = 1$.

Substituting these components into Minkowski's Inequality gives:

$$\sqrt{a^2+1}+\sqrt{b^2+1}+\sqrt{c^2+1} \geq \sqrt{(a+b+c)^2+(1+1+1)^2} = \sqrt{(a+b+c)^2+9}$$

Equality holds if and only if the vectors $(a, 1)$, $(b, 1)$, and $(c, 1)$ are proportional:

$$\frac{a}{1} = \frac{b}{1} = \frac{c}{1} \implies a = b = c$$

Thus, equality holds if and only if $a = b = c$.