ROMANIAN MATHEMATICAL MAGAZINE

SP.609 If $a, b, c > 0$ then:

$$\sqrt{a^2b^2+c^2}+\sqrt{c^2b^2+a^2}+\sqrt{c^2a^2+b^2}\geq\sqrt{(a+b+c)^2+(ab+bc+ca)^2}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

Let be $f:(0,\infty)\to\mathbb{R}; f(x)=\sqrt{x^2+1}$

$f'(x)=\frac{x}{\sqrt{x^2+1}}; f''(x)=\frac{1}{(x^2+1)\sqrt{x^2+1}}>0$; f convex

By Jensen's inequality:

$$\frac{\sum_{cyc}abf\left(\frac{c}{ab}\right)}{\sum_{cyc}ab}\geq f\left(\frac{\sum_{cyc}ab\cdot\frac{c}{ab}}{\sum_{cyc}ab}\right)=f\left(\frac{\sum_{cyc}c}{\sum_{cyc}ab}\right)$$

$$\frac{\sum_{cyc}abf\left(\frac{c}{ab}\right)}{\sum_{cyc}ab}\geq f\left(\frac{a+b+c}{ab+bc+ca}\right)$$

$$abf\left(\frac{c}{ab}\right)+bcf\left(\frac{a}{bc}\right)+caf\left(\frac{b}{ca}\right)\geq(ab+bc+ca)f\left(\frac{a+b+c}{ab+bc+ca}\right)$$

$$\sum_{cyc}ab\sqrt{\left(\frac{c}{ab}\right)^2+1}\geq(ab+bc+ca)\sqrt{\left(\frac{a+b+c}{ab+bc+ca}\right)^2+1}$$

$$\sum_{cyc}\sqrt{c^2+a^2b^2}\geq\sqrt{(a+b+c)^2+(ab+bc+ca)^2}$$

Equality holds for $a=b=c=1$.

Solution 2 by Jose Luis Diaz – Barrero – Spain

Minkowski's Inequality (the triangle inequality for vectors) states that for any real numbers:

$$\sqrt{x_1^2+y_1^2}+\sqrt{x_2^2+y_2^2}+\sqrt{x_3^2+y_3^2}\geq\sqrt{(x_1+x_2+x_3)^2+(y_1+y_2+y_3)^2}$$

We can strategically choose the components (x_i, y_i) from each radical term as follows:

- For $\sqrt{a^2b^2+c^2}$, let $x_1=ab$ and $y_1=c$.
- For $\sqrt{b^2c^2+a^2}$, let $x_2=bc$ and $y_2=a$.

- For $\sqrt{c^2a^2+b^2}$, let $x_3 = ca$ and $y_3 = b$.

Applying these to Minkowski's Inequality yields:

$$\sqrt{(ab)^2+c^2}+\sqrt{(bc)^2+a^2}+\sqrt{(ca)^2+b^2} \geq \sqrt{(ab+bc+ca)^2+(c+a+b)^2}$$

Equality holds if and only if the vectors $(ab, c), (bc, a)$, and (ca, b) are linearly dependent (proportional):

$$\frac{ab}{bc} = \frac{c}{a} \Rightarrow \frac{a}{c} = \frac{c}{a} \Rightarrow a^2 = c^2$$

Since $a, b, c > 0$, this implies $a = c$. By symmetry across the other ratios, equality holds if and only if $a = b = c$.

Solution 3 by Omkumar Rao-India

$$\vec{u} = (ab, c), \vec{v} = (bc, a), , \vec{w} = (ca, b) \in R^2:$$

$$|\vec{u}| = \sqrt{a^2b^2+c^2}, |\vec{v}| = \sqrt{c^2b^2+a^2}$$
$$|\vec{w}| = \sqrt{c^2a^2+b^2}$$

$$Minkowski\ Inequality: |\vec{u}|+|\vec{v}|+|\vec{w}| \geq |\vec{u}+\vec{v}+\vec{w}|$$

$$\vec{u}+\vec{v}+\vec{w} = (ab+bc+ca, a+b+c):$$
$$\sqrt{a^2b^2+c^2}+\sqrt{c^2b^2+a^2}+\sqrt{c^2a^2+b^2} \geq \sqrt{(a+b+c)^2+(ab+bc+ca)^2}$$

$$Equality: \vec{u}||\vec{v}||\vec{w} \Leftrightarrow \frac{ab}{c} = \frac{bc}{a} = \frac{ca}{b} \Leftrightarrow a = b = c$$