José Luis **Díaz-Barrero**
Barcelona Mathematical Circle (BMC)
jose.luis.diaz@upc.edu

SP.609. *Proposed by Daniel Sitaru, Romania.* If $a, b, c > 0$ then

$$\sqrt{a^2b^2 + c^2} + \sqrt{b^2c^2 + a^2} + \sqrt{c^2a^2 + b^2} \geq \sqrt{(a + b + c)^2 + (ab + bc + ca)^2}.$$

Solution by José Luis Díaz-Barrero, Barcelona, Spain. Minkowski's Inequality (the triangle inequality for vectors) states that for any real numbers:

$$\sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2} + \sqrt{x_3^2 + y_3^2} \geq \sqrt{(x_1 + x_2 + x_3)^2 + (y_1 + y_2 + y_3)^2}$$

We can strategically choose the components (x_i, y_i) from each radical term as follows:

- For $\sqrt{a^2b^2 + c^2}$, let $x_1 = ab$ and $y_1 = c$.
- For $\sqrt{b^2c^2 + a^2}$, let $x_2 = bc$ and $y_2 = a$.
- For $\sqrt{c^2a^2 + b^2}$, let $x_3 = ca$ and $y_3 = b$.

Applying these to Minkowski's Inequality yields:

$$\sqrt{(ab)^2 + c^2} + \sqrt{(bc)^2 + a^2} + \sqrt{(ca)^2 + b^2} \geq \sqrt{(ab + bc + ca)^2 + (c + a + b)^2}$$

Equality holds if and only if the vectors (ab, c), (bc, a), and (ca, b) are linearly dependent (proportional):

$$\frac{ab}{bc} = \frac{c}{a} \Rightarrow \frac{a}{c} = \frac{c}{a} \Rightarrow a^2 = c^2$$

Since $a, b, c > 0$, this implies $a = c$. By symmetry across the other ratios, equality holds if and only if $a = b = c$.