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SP.608 If $a, b, c \in (0, 1)$ and $x, y, z > 0$ such that $a = (bc)^x, b = (ca)^y, c = (ab)^z$ and $xyz = 1$ then holds:

$$\sqrt[n]{\sum_{cyc} a^n (y+z+2)^{2n-1}} \geq 6 \cdot \sqrt[3]{abc}, n \in \mathbb{N}^*, n \geq 2$$

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Solution by proposers

$$\begin{aligned} \sum_{cyc} a^n (y+z+2)^{2n-1} &= \sum_{cyc} \frac{a^n (y+z+2)^n}{\frac{1}{(y+z+2)^{n-1}}} \stackrel{\text{Radon}}{\geq} \\ &\geq \frac{[a(y+z+2) + b(z+x+2) + x(x+y+2)]^n}{\left(\frac{1}{x+y+2} + \frac{1}{y+z+2} + \frac{1}{x+y+2}\right)^{n-1}} = \\ &= \frac{[(b+c)x + (c+a)y + (a+b)x]^n}{\left(\frac{1}{x+y+2} + \frac{1}{y+z+2} + \frac{1}{x+y+2}\right)^{n-1}} \stackrel{\text{AM-GM}}{\geq} \\ &\geq \frac{(3\sqrt[3]{(a+b)(b+c)(c+a)xyz})^n}{\left(\frac{1}{x+y+2} + \frac{1}{y+z+2} + \frac{1}{x+y+2}\right)^{n-1}} \geq \frac{(6\sqrt[3]{abc})^n}{\left(\frac{1}{x+y+2} + \frac{1}{y+z+2} + \frac{1}{x+y+2}\right)^{n-1}} \quad (1) \end{aligned}$$

Now, from $a = (bc)^x, b = (ca)^y, c = (ab)^z$, we get $x = \frac{-\ln a}{-\ln b - \ln c}, y = \frac{-\ln b}{-\ln c - \ln a}$

and $z = \frac{-\ln c}{-\ln a - \ln b}$. Let us denote $\begin{cases} -\ln a - \ln c = q \\ -\ln b - \ln c = p \\ -\ln a - \ln b = r \end{cases}$ with $p, q, r > 0$, then

$$\begin{cases} x = \frac{-p+q+r}{2p} \\ y = \frac{p-q+r}{2q} \\ z = \frac{p+q-r}{2r} \end{cases} \text{ and using } \frac{pq}{p+q} \leq \frac{p+q}{2}, \text{ we get: } \frac{1}{x+y+2} = \frac{2pq}{(p+q+r)(p+q)} \leq \frac{p+q}{2(p+q+r)} \quad (\text{and analogs})$$

Hence, it follows that: $\frac{1}{x+y+2} + \frac{1}{y+z+2} + \frac{1}{z+x+2} \leq \frac{p+q+q+r+r+p}{2(p+q+r)} = 1 \quad (2)$

From (1) and (2) we obtain:

$$a^n (y+z+2)^{2n-1} + b^n (z+x+2)^{2n-1} + c^n (x+y+z)^{2n-1} \geq (6\sqrt[3]{abc})^n$$

Finally,

$$\sqrt[n]{\sum_{cyc} a^n (y+z+2)^{2n-1}} \geq 6 \cdot \sqrt[3]{abc}, n \in \mathbb{N}^*, n \geq 2$$