

ROMANIAN MATHEMATICAL MAGAZINE

SP.607 Let be the function $f: [0, 1] \rightarrow \mathbb{R}$ integrable such that $f(1) = 1$ and

$$\int_x^y f(t) dt = \frac{1}{2} (yf(y) - xf(x)), \forall x, y \in [0, 1]$$

Find:

$$I = \int_0^{\frac{\pi}{4}} f(x) \cdot \tan^2 x dx$$

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Solution by proposers

$$\int_x^y f(t) dt = \frac{1}{2} (yf(y) - xf(x)), \forall x, y \in [0, 1] \quad (*)$$

If in (*) we take $x = 0$ and $y = x$, we get $\int_0^x f(t) dt = \frac{x}{2} f(x)$.

Let be $F(x) = \int_0^x f(t) dt$, how f is integrable function then f is bounded function and

$$F(x) = \frac{x}{2} f(x) \quad (1)$$

$$\text{From } F'(x) = f(x) \Rightarrow xF'(x) = 2F(x) \quad (2)$$

$$\text{On the other hand, } \left(\frac{F(x)}{x^2} \right)' = \frac{xF'(x) - 2F(x)}{x^3} \stackrel{(2)}{=} 0 \Rightarrow F(x) = \alpha x^2, \alpha \in \mathbb{R} \Rightarrow$$

$f(x) = \beta(x), \beta \in \mathbb{R}, x \in (0, 1]$ and from $f(1) = 1$, we get $\beta = 1$ and hence $f(x) = x$.

Therefore,

$$\begin{aligned} I &= \int_0^{\frac{\pi}{4}} f(x) \cdot \tan^2 x dx = \int_0^{\frac{\pi}{4}} \frac{x(1 - \cos^2 x)}{\cos^2 x} dx = \\ &= \int_0^{\frac{\pi}{4}} \frac{x}{\cos^2 x} dx - \int_0^{\frac{\pi}{4}} x dx = \left(x \tan x - \frac{\pi}{2} \right) \Big|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan x dx = \frac{\pi}{4} - \frac{\pi^2}{32} - \frac{1}{2} \ln 2. \end{aligned}$$