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SP.606. *Proposed by Daniel Sitaru, Romania.* Let $A(2,1,0); B(0,1,2); C(3,0,1); D(4,4,4)$ be four points in $\mathbb{R}^3$. Find the volume of the tetrahedron $ABCD$.

Solution by José Luis Díaz-Barrero, Barcelona, Spain. To find the volume of the tetrahedron with vertices $A(2,1,0)$, $B(0,1,2)$, $C(3,0,1)$, and $D(4,4,4)$, we use the scalar triple product of three vectors sharing a common vertex. The formula for the volume V is:

$$V = \frac{1}{6}|(\overrightarrow{AB} \times \overrightarrow{AC}) \cdot \overrightarrow{AD}|$$

Choosing point A as the reference vertex, we subtract its coordinates from the other points:

$$\begin{aligned}
\overrightarrow{AB} &= (0-2,\ 1-1,\ 2-0) = (-2,\ 0,\ 2)\\
\overrightarrow{AC} &= (3-2,\ 0-1,\ 1-0) = (1,\ -1,\ 1)\\
\overrightarrow{AD} &= (4-2,\ 4-1,\ 4-0) = (2,\ 3,\ 4)
\end{aligned}$$

The scalar triple product corresponds to the determinant of the matrix formed by these three vectors:

$$D = \det \begin{pmatrix} -2 & 0 & 2 \\ 1 & -1 & 1 \\ 2 & 3 & 4 \end{pmatrix}$$

Expanding along the first row, we get

$$\begin{aligned}
D &= -2 \cdot \det \begin{pmatrix} -1 & 1 \\ 3 & 4 \end{pmatrix} - 0 \cdot \det \begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix} + 2 \cdot \det \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}\\
&= -2((-1)(4) - (1)(3)) - 0 + 2((1)(3) - (-1)(2)) = 24
\end{aligned}$$

Plugging the absolute value of the determinant into our volume formula, we obtain

$$V = \frac{1}{6} \cdot |24| = \frac{24}{6} = 4$$