SP.606 Let be $A(2,1,0)$; $B(0,1,2)$; $C(3,0,1)$; $D(4,4,4)$. Find the volume of the tetrahedron $ABCD$.

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$\overrightarrow{AB} = (x_B - x_A)\vec{i} + (y_B - y_A)\vec{j} + (z_B - z_A)\vec{k}$$

$$\overrightarrow{AB} = (0-2)\vec{i} + (1-1)\vec{j} + (2-0)\vec{k} = -2\vec{i} + 2\vec{k}$$

$$\overrightarrow{AC} = (x_i - x_A)\vec{i} + (y_C - y_A)\vec{j} + (z_C - z_A)\vec{k}$$

$$\overrightarrow{AC} = (3-2)\vec{i} + (0-1)\vec{j} + (1-0)\vec{k} = \vec{i} - \vec{j} + \vec{k}$$

$$\overrightarrow{AD} = (x_D - x_A)\vec{i} + (y_D - y_A)\vec{j} + (z_D - z_A)\vec{k}$$

$$\overrightarrow{AD} = (4-2)\vec{i} + (4-1)\vec{j} + (4-0)\vec{j} = 2\vec{i} + 3\vec{j} + 4\vec{k}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 0 & 2 \\ 1 & -1 & 1 \end{vmatrix} = 2\vec{j} + 2\vec{k} + 2\vec{i} + 2\vec{j}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = 2\vec{i} + 4\vec{j} + 2\vec{k}$$

$$\overrightarrow{AD} \cdot \left(\overrightarrow{AB} \times \overrightarrow{AC}\right) = 2 \cdot 2 + 3 \cdot 4 + 4 \cdot 2 = 4 + 12 + 8 = 24$$

$$V[ABCD] = \frac{1}{6}\left|\overrightarrow{AD} \cdot \left(\overrightarrow{AB} \times \overrightarrow{AC}\right)\right| = \frac{24}{6} = 4$$

Solution 2 by Marin Chirciu-Romania

$$V[ABCD] = \frac{1}{6}|\Delta|, \quad \Delta = \begin{vmatrix} x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \\ x_D & y_D & z_D & 1 \end{vmatrix}.$$

$$\Delta = \begin{vmatrix} 2 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 3 & 0 & 1 & 1 \\ 4 & 4 & 4 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 0 & 1 \\ -2 & 0 & 2 & 0 \\ 1 & -1 & 1 & 0 \\ 2 & 3 & 4 & 0 \end{vmatrix} = -\begin{vmatrix} -2 & 0 & 2 \\ 1 & -1 & 1 \\ 2 & 3 & 4 \end{vmatrix} = 2\begin{vmatrix} -1 & 2 \\ 3 & 6 \end{vmatrix} = -24.$$

$$V[ABCD] = \frac{1}{6}|\Delta| = \frac{1}{6}|-24| = 4.$$