ROMANIAN MATHEMATICAL MAGAZINE

SP.605 Let be $A(1,2,1); B(2,1,3); C(4,4,4)$. Find the distance from the point A to the line BC.

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$\overrightarrow{AB} = (x_B - x_A)\vec{i} + (y_B - y_A)\vec{j} + (z_B - z_A)\vec{k} =$$

$$= (2-1)\vec{i} + (1-2)\vec{j} + (3-1)\vec{k} = \vec{i} - \vec{j} + 2\vec{k}$$

$$\overrightarrow{AC} = (x_C - x_A)\vec{i} + (y_C - y_A)\vec{j} + (z_C - z_A)\vec{k} =$$

$$= (4-1)\vec{i} + (4-2)\vec{j} + (4-1)\vec{k} = 3\vec{i} + 2\vec{j} + 3\vec{k}$$

$$BC = \sqrt{(x_C - x_B)^2 + (y_C - y_B)^2 + (z_C - z_B)^2} =$$

$$= \sqrt{(4-2)^2 + (4-1)^2 + (4-3)^2} = \sqrt{4+9+1} = \sqrt{14}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 2 \\ 3 & 2 & 3 \end{vmatrix} = -3\vec{i} + 6\vec{j} + 2\vec{k} + 3\vec{k} - 4\vec{i} - 3\vec{j} = -7\vec{i} + 3\vec{j} + 5\vec{k}$$

$$A[ABC] = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\sqrt{(-7)^2 + 3^2 + 5^2} =$$

$$= \frac{1}{2}\sqrt{49+9+25} = \frac{1}{2}\sqrt{58+25} = \frac{1}{2}\sqrt{83}$$

$$d(A,B,C) = \frac{2[ABC]}{BC} = \frac{2 \cdot \frac{1}{2}\sqrt{83}}{\sqrt{14}} = \sqrt{\frac{83}{14}}$$

Solution 2 by Marin Chirciu-Romania

The distance from the point $A(x_1, y_1, z_1)$ **to the line** BC, $B(x_2, y_2, z_2), C(x_3, y_3, z_3)$ **is:**

$$d(A, BC) = \frac{\sqrt{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix}^2 + \begin{vmatrix} y_2 - y_1 & z_2 - z_1 \\ y_3 - y_1 & z_3 - z_1 \end{vmatrix}^2 + \begin{vmatrix} z_2 - z_1 & x_2 - x_1 \\ z_3 - z_1 & x_3 - x_1 \end{vmatrix}^2}}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}.$$

The distance from the point $A(1,2,1)$ **to the line** BC, $B(2,1,3); C(4,4,4)$ **is:**

$$d(A, BC) = \frac{\sqrt{\begin{vmatrix} 2-1 & 1-2 \\ 4-1 & 4-2 \end{vmatrix}^2 + \begin{vmatrix} 1-2 & 3-1 \\ 4-2 & 4-1 \end{vmatrix}^2 + \begin{vmatrix} 3-1 & 2-1 \\ 4-1 & 4-1 \end{vmatrix}^2}}{\sqrt{(4-2)^2 + (4-1)^2 + (4-3)^2}} =$$

$$= \frac{\sqrt{\begin{vmatrix}1 & -1\\ 3 & 2\end{vmatrix}^2 + \begin{vmatrix}-1 & 2\\ 2 & 3\end{vmatrix}^2 + \begin{vmatrix}2 & 1\\ 3 & 3\end{vmatrix}^2}}{\sqrt{4+9+1}} = \sqrt{\frac{83}{14}}$$

Solution 3 by Jose Luis Diaz Barrero-Spain

To find the distance from point $A(1,2,1)$ to the line passing through $B(2,1,3)$ and $C(4,,4,4)$ we can use the vector formula:

$$d = \frac{\left|\left|\overrightarrow{BA} \times \overrightarrow{BC}\right|\right|}{\left|\left|\overrightarrow{BC}\right|\right|}$$

We find the components by subtracting the initial point from the terminal point:

$$\overrightarrow{BA} = (1-2, 2-1, 1-3) = (-1, 1, -2)$$

$$\overrightarrow{BC} = (4-2, 4-1, 4-3) = (2, 3, 1)$$

Using the determinant definition of the cross product:

$$\overrightarrow{BA} \times \overrightarrow{BC} = \det\begin{pmatrix} i & j & k \\ -1 & 1 & -2 \\ 2 & 3 & 1 \end{pmatrix}$$

Expanding along the first row:

$$\overrightarrow{BA} \times \overrightarrow{BC} = i(1\cdot 1 - (-2)\cdot 3) - j\big((-1)\cdot 1 - (-2)\cdot 2\big) + k\big((-1)\cdot 3 - 2\cdot 2\big)$$

$$= i(1+6) - j(-1+4) + k(-3-2)$$

$$= (7, -3, -5)$$

Next, we find the lengths of the cross product vector and the direction vector $\overrightarrow{BC}$:

$$\left|\left|\overrightarrow{BA} \times \overrightarrow{BC}\right|\right| = \sqrt{7^2 + (-3)^2 + (-5)^2} = \sqrt{49+9+25} = \sqrt{83}$$

$$\left|\left|\overrightarrow{BC}\right|\right| = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{4+9+1} = \sqrt{14}$$

Substituting these values back into the distance formula gives:

$$d = \frac{\sqrt{83}}{\sqrt{14}} = \sqrt{\frac{83}{14}}$$

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