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SP.605. *Proposed by Daniel Sitaru, Romania.* Let be $A(1,2,1); B(2,1,3); C(4,4,4)$ be three points in $\mathbb{R}^3$. Find the distance from the point A to the line BC.

Solution by José Luis Díaz-Barrero, Barcelona, Spain. To find the distance from point $A(1,2,1)$ to the line passing through $B(2,1,3)$ and $C(4,4,4)$, we can use the vector formula:

$$d = \frac{\|\overrightarrow{BA} \times \overrightarrow{BC}\|}{\|\overrightarrow{BC}\|}$$

We find the components by subtracting the initial point from the terminal point:

$$\begin{aligned} \overrightarrow{BA} &= (1-2,\ 2-1,\ 1-3) = (-1,\ 1,\ -2) \\ \overrightarrow{BC} &= (4-2,\ 4-1,\ 4-3) = (2,\ 3,\ 1) \end{aligned}$$

Using the determinant definition of the cross product:

$$\overrightarrow{BA} \times \overrightarrow{BC} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & -2 \\ 2 & 3 & 1 \end{pmatrix}$$

Expanding along the first row:

$$\begin{aligned} \overrightarrow{BA} \times \overrightarrow{BC} &= \mathbf{i}(1 \cdot 1 - (-2) \cdot 3) - \mathbf{j}((-1) \cdot 1 - (-2) \cdot 2) + \mathbf{k}((-1) \cdot 3 - 1 \cdot 2) \\ &= \mathbf{i}(1+6) - \mathbf{j}(-1+4) + \mathbf{k}(-3-2) \\ &= (7,\ -3,\ -5) \end{aligned}$$

Next, we find the lengths of the cross product vector and the direction vector $\overrightarrow{BC}$:

$$\begin{aligned} \|\overrightarrow{BA} \times \overrightarrow{BC}\| &= \sqrt{7^2 + (-3)^2 + (-5)^2} = \sqrt{49+9+25} = \sqrt{83} \\ \|\overrightarrow{BC}\| &= \sqrt{2^2 + 3^2 + 1^2} = \sqrt{4+9+1} = \sqrt{14} \end{aligned}$$

Substituting these values back into the distance formula gives:

$$d = \frac{\sqrt{83}}{\sqrt{14}} = \sqrt{\frac{83}{14}}$$