SP.604 Let be $A(2,1,0); B(1,2,1); C(3,3,3); D(0,0,4)$. Find the distance from the point D to the real plan (ABC).

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$(ABC):\begin{vmatrix} x & y & z & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 \\ 3 & 3 & 3 & 1 \end{vmatrix}=0$$

$$(ABC):\begin{vmatrix} x-3 & y-3 & z-3 & 0 \\ -1 & -2 & -3 & 0 \\ -2 & -1 & -2 & 0 \\ 3 & 3 & 3 & 1 \end{vmatrix}=0$$

$$(ABC):\begin{vmatrix} x-3 & y-3 & z-3 \\ -1 & -2 & -3 \\ -2 & -1 & -2 \end{vmatrix}=0$$

$$(ABC):\begin{vmatrix} x-3 & y-3 & z-3 \\ 1 & 2 & 3 \\ 2 & 1 & 2 \end{vmatrix}=0$$

$$(ABC): 4(x-3)+6(y-3)+(z-3)-4(z-3)-3(x-3)-2(y-3)=0$$

$$(ABC): x-3+4(y-3)-3(z-3)=0$$

$$(ABC): x+4y-3z-3-12+9=0$$

$$(ABC): x+4y-3z-6=0$$

$$d(D,(ABC))=\frac{|x_D+4y_D-3z_D-6|}{\sqrt{1^2+4^2+(-3)^2}}=\frac{|0+4\cdot 0-3\cdot 4-6|}{\sqrt{1^2+4^2+(-3)^2}}$$

$$d(D,(ABC))=\frac{18}{\sqrt{26}}=\frac{18\sqrt{26}}{26}=\frac{9\sqrt{26}}{13}$$

Solution 2 by Marin Chirciu-Romania

The equation of the real plan $(M_1M_2M_3)$, $M_1(x_1,y_1,z_1); M_2(x_2,y_2,z_2); M_3(x_3,y_3,z_3)$ is:

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0.$$

2).The distance from $M_0(x_0,y_0,z_0)$ to $(P): ax+by+cx+d=0$ is:

$$d\left(M_0,(P)\right)=\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}.$$

$$(ABC):\begin{vmatrix} x-2 & y-1 & z-0 \\ 1-2 & 2-1 & 1-0 \\ 3-2 & 3-1 & 3-0 \end{vmatrix}=0 \Leftrightarrow x+4y-3z-6=0.$$

The distance from $D(0,0,4)$ to $(ABC): x+4y-3z-6=0$ is:

$$d\left(D,(ABC)\right)=\frac{|1\cdot 0+4\cdot 0-3\cdot 4-6|}{\sqrt{1^2+4^2+(-3)^2}}=\frac{18}{\sqrt{26}}.$$

Solution 3 by José Luis Díaz – Barrero – Spain

We can find two vectors lying in the plane by using the points A, B, and C:

$$\overrightarrow{AB}=B-A=(1-2,2-1,1-0)=(-1,1,1)$$
$$\overrightarrow{AC}=C-A=(3-2,3-1,3-0)=(1,2,3)$$

The normal vector $\vec{n}$ to the plane is perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$, which can be found using the cross product:

$$\vec{n}=\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

$$\vec{n}=\vec{\imath}(1\cdot 3-1\cdot 2)-\vec{\jmath}\big((-1)\cdot 3-1\cdot 1\big)+\vec{k}\big((-1)\cdot 2-1\cdot 1\big)$$

$$\vec{n}=1\vec{\imath}+4\vec{\jmath}-3\vec{k}=(1,4,-3)$$

Using the point-normal equation of a plane $A(x-x_0)+B(y-y_0)+C(z-z_0)=0$ with the normal vector $\vec{n}=(1,4,-3)$ and the point $A(2,1,0)$, we get

$$1(x-2)+4(y-1)-3(z-0)=0 \Leftrightarrow x+4y-3z-6=0$$

The formula for the distance d from a point (x_0,y_0,z_0) to a plane $Ax+By+Cz+D=0$ is:

$$d=\frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}}$$

Substitute the plane coefficients $(A=1, B=4, C=-3, D=-6)$ and coordinates of $D(0,0,4)$, to obtain

$$d=\frac{|1(0)+4(0)-3(4)-6|}{\sqrt{1^2+4^2+(-3)^2}}=\frac{18}{\sqrt{26}}=\frac{9\sqrt{26}}{13},$$

and the distance from the point D to the plane (ABC) is $\frac{9\sqrt{26}}{13}$.