ROMANIAN MATHEMATICAL MAGAZINE

SP.603 Prove that $\frac{5}{3}$ is the largest positive value of the power k such that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\geq a^k+b^k+c^k+d^k$$

for all positive real numbers a,b,c,d with at most one of them smaller than 1 and $ab+ac+ad+bc+bd+cd=6$.

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Solution by proposer

For $a\in\left[1,\frac{5}{2}\right),b=c=1$ and $d=\frac{5-2a}{a+2}$, the constraints are satisfied and the inequality becomes $E(a)\geq 0$, where

$$E(a)=\frac{1}{a}+\frac{a+2}{5-2a}-a^k-\left(\frac{5-2a}{a+2}\right)^k.$$

We have

$$E'(a)=-\frac{1}{a^2}+\frac{9}{(5-2a)^2}-ka^{k-1}+\frac{9k}{(a+2)^2}d^{k-1},$$

$$E''(a)=\frac{2}{a^3}+\frac{36}{(5-2a)^3}-k(k-1)a^{k-2}-\frac{18k}{(a+2)^3}d^{k-1}-\frac{81k(k-1)}{(a+2)^4}d^{k-2}$$

Since $E(1)=E'(1)=0$, the condition $E''(1)\geq 0$ is necessary to have $E(a)\geq 0$ for $a\in\left[1,\frac{5}{2}\right)$. From

$$E''(1)=2+\frac{4}{3}-k(k-1)-\frac{2k}{3}-k(k-1)=\frac{2(5-3k)(1+k)}{3}\geq 0,$$

we get $k\leq\frac{5}{3}$. To show that $\frac{5}{3}$ is the largest value of k, we need to prove that $F\geq 0$ for $a\geq b\geq c\geq 1\geq d$ such that $ab+ac+ad+bc+bd+cd=6$, where

$$F=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}-a^{\frac{5}{3}}-b^{\frac{5}{3}}-c^{\frac{5}{3}}-d^{\frac{5}{3}}.$$

For fixed a and b, assume that d and F are functions of c. We have

$$(a+b+c)d'+a+b+d=0$$

and

$$F'(c) = -c^{-2} - \frac{5}{3}c^{\frac{2}{3}} - \left(d^{-2} + \frac{5}{3}d^{\frac{2}{3}}\right)d' = -c^{-2} - \frac{5}{3}c^{\frac{2}{3}} + \left(d^{-2} + \frac{5}{3}d^{\frac{2}{3}}\right)\frac{a+b+d}{a+b+c}.$$

We claim that $F'(c) \geq 0$, that is

$$\frac{3d^{-2} + 5d^{\frac{2}{3}}}{3c^{-2} + 5c^{\frac{2}{3}}} \geq \frac{a+b+c}{a+b+d}, \quad \frac{3d^{-2} + 5d^{\frac{2}{3}}}{3c^{-2} + 5c^{\frac{2}{3}}} - 1 \geq \frac{a+b+c}{a+b+d} - 1,$$

$$\frac{3(c^2 - d^2) - 5c^2d^2\left(c^{\frac{2}{3}} - d^{\frac{2}{3}}\right)}{d^2\left(3 + 5c^{\frac{8}{3}}\right)} \geq \frac{c-d}{a+b+d}.$$

From : $6 = ab + ac + ad + bc + bd + cd \geq 6\sqrt{abcd} \geq 6\sqrt{c^3d}$,

we have $c^3d \leq 1$, hence

$$d^2\left(3 + 5c^{\frac{8}{3}}\right) - 8cd = -3d(c-d) - 5cd\left(1 - c^{\frac{5}{3}}d\right) \leq 0.$$

Since $cd \leq 1, d^2\left(3 + 5c^{\frac{8}{3}}\right) \leq 8cd$ and $a + b \geq 2c$, it suffices to prove the homogeneous inequality

$$\frac{3(c^2 - d^2) - 5(cd)^{\frac{2}{3}}\left(c^{\frac{2}{3}} - d^{\frac{2}{3}}\right)}{8cd} \geq \frac{c-d}{2c+d}.$$

For $c = x^3 \geq 1$ and $d = 1$, the inequality becomes

$$\frac{3(x^6 - 1) - 5x^2(x^2 - 1)}{8x^3} \geq \frac{x^3 - 1}{2x^3 + 1}, \quad (x-1)(A-B) \geq 0,$$

where

$$A = (2x^3 + 1)\left(3x^5 + 3x^4 - 2x^3 - 2x^2 + 3x + 3\right), \ \ B = 8x^3(x^2 + x + 1).$$

Since: $A - B = 6x^8 + 6x^7 - 4x^6 - 9x^5 + x^4 - 4x^3 - 2x^2 + 3x + 3 =$

$$= (x-1)\left(6x^7 + 12x^6 + 8x^5 - x^4 - 4x^2 - 6x - 3\right) \geq 0,$$

we have $F'(c) \geq 0, F(c)$ is increasing and has the minimum value when c is minimum, hence when $c = 1$. So, it suffices to consider this case, when we need to show that $G > 0$ for $a \geq b \geq 1 \geq d$ such that $ab + bd + da + a + b + d = 6$, where

$$G = \frac{1}{a} + \frac{1}{b} + \frac{1}{d} - a^{\frac{5}{3}} - b^{\frac{5}{3}} - d^{\frac{5}{3}}.$$

For fixed a, assume that d and G are functions of b. We have

$$(a+b+1)d'+a+d+1=0$$

and

$$G'(b)=-\frac{1}{b^2}-\frac{5}{3}b^{\frac{2}{3}}-\left(\frac{1}{d^2}+\frac{5}{3}d^{\frac{2}{3}}\right)d'=-\frac{1}{b^2}-\frac{5}{3}b^{\frac{2}{3}}+\left(\frac{1}{d^2}+\frac{5}{3}d^{\frac{2}{3}}\right)\frac{a+d+1}{a+b+1}.$$

We claim that $G'(b)\ge 0$, that is

$$\frac{3d^{-2}+5d^{\frac{2}{3}}}{3b^{-2}+5b^{\frac{2}{3}}}\ge\frac{a+b+1}{a+d+1},\qquad\frac{3d^{-2}+5d^{\frac{2}{3}}}{3b^{-2}+5b^{\frac{2}{3}}}-1\ge\frac{a+b+1}{a+d+1}-1,$$

$$\frac{3(b^2-d^2)-5b^2d^2\left(b^{\frac{2}{3}}-d^{\frac{2}{3}}\right)}{d^2\left(3+5b^{\frac{8}{3}}\right)}\ge\frac{b-d}{a+d+1}.$$

From: $6=ab+bd+da+a+b+d\ge 6\sqrt{abd}\ge 6\sqrt{b^2d}$,

we get $b^2d\le 1$, hence

$$d^2\left(3+5b^{\frac{8}{3}}\right)-8bd=-3d(b-d)-5bd\left(1-b^{\frac{5}{3}}d\right)\le 0,$$

and from

$$6=ab+bd+da+a+b+d\ge b^2+2bd+2b+d=$$

$$=(b+d)^2+2(b+d)-d^2-d\ge(b+d)^2+2(b+d)-2,$$

we get

$$(b+d)^2+2(b+d)-8\le 0,\ \ b+d\le 2,$$

hence

$$a+d+1\ge b+d=1\ge b+d+\frac{b+d}{2}=\frac{3(b+d)}{2}.$$

Since $bd\le 1, d^2\left(3+5b^{\frac{8}{3}}\right)\le 8bd$ and $a+d+1\ge b+d+1\ge b+d+\frac{b+d}{2}=\frac{3(b+d)}{2}$, it suffices to prove the homogeneous inequality

$$\frac{3(b^2-d^2)-5(bd)^{\frac{2}{3}}\left(b^{\frac{2}{3}}-d^{\frac{2}{3}}\right)}{8bd}\ge\frac{2(b-d)}{3(b+d)}.$$

For $b=x^3\ge 1$ and $d=1$, the inequality becomes

$$\frac{3(x^6-1)-5x^2(x^2-1)}{8x^3}\ge\frac{2(x^3-1)}{3(x^3+1)},\ \ (x-1)(C-D)\ge 0,$$

where

$$C = 3(x^3+1)(3x^5+3x^4-2x^3-2x^2+3x+3), \qquad D = 16x^3(x^2+x+1).$$

Since

$$C - D = 9x^8+9x^7-6x^6-13x^5+2x^4-13x^3-6x^2+9x+9$$
$$= (x-1)^2(9x^6+27x^5+39x^4+38x^3+39x^2+27x+9) \geq 0,$$

we have $G'(b) \geq 0, G(b)$ is increasing and has the minimum value when b is minimum, hence when $b = 1$. So, it suffices to consider this case, when we need to show that $H \geq 0$ for $a \geq 1 \geq d$ such that $ad + 2(a+d) = 5$, where

$$G = \frac{1}{a} + \frac{1}{d} - a^{\frac{5}{3}} - d^{\frac{5}{3}}.$$

We have

$$d = \frac{5-2a}{a+2}, a < \frac{5}{2}.$$

Assume that d and G are function of a, and show that $G'(a) \geq 0$. Then, $G(a)$ is increasing and $G(a) \geq G(1) = 0$ (because $a = 1$ implies $d = 1$). We have

$$(a+2)d' + d + 2 = 0$$

and

$$G'(a) = -a^{-2} - \frac{5}{3}a^{\frac{2}{3}} - \left(d^{-2} + \frac{5}{3}d^{\frac{2}{3}}\right)d' = -a^{-2} - \frac{5}{3}a^{\frac{2}{3}} + \left(d^{-2} + \frac{5}{3}d^{\frac{2}{3}}\right)\frac{d+2}{a+2}.$$

Write the required inequality $G'(a) \geq 0$ as follows:

$$\frac{3d^{-2}+5d^{\frac{2}{3}}}{3a^{-2}+5a^{\frac{2}{3}}} \geq \frac{a+2}{d+2}, \quad \frac{3d^{-2}+5d^{\frac{2}{3}}}{3a^{-2}+5a^{\frac{2}{3}}} - 1 \geq \frac{a+2}{d+2} - 1,$$

$$\frac{3(a^2-d^2) - 5a^2d^2\left(a^{\frac{2}{3}} - d^{\frac{2}{3}}\right)}{d^2\left(3+5a^{\frac{8}{3}}\right)} \geq \frac{a-d}{d+2}.$$

We will show that this inequality can be obtained by multiplying the inequality

$$8(d+2) \geq 3d^2\left(3+5a^{\frac{8}{3}}\right) \qquad (*)$$

and $3(a^2-d^2) - 5a^2d^2\left(a^{\frac{2}{3}} - d^{\frac{2}{3}}\right) \geq \frac{8(a-d)}{3}$ (**)

By Bernoulli's inequality, we have

$$a^{\frac{8}{3}} = [1+(a^3-1)]^{\frac{8}{9}} \le 1+\frac{8(a^3-1)}{9} = \frac{8a^3+1}{9}, \quad 3+5a^{\frac{8}{3}} \le \frac{8(5a^3+4)}{9}.$$

So, to prove the inequality (*), it suffices to show that: $3(d+2) \ge d^2(5a^3+4)$,

which is equivalent to

$$\frac{3(5-2a)}{a+2}+6 \ge \frac{(5-2a)^2(5a^3+4)}{(a+2)^2},$$

$$-20a^5+100a^4-125a^3-16a^2+107a-46 \ge 0, \quad (a-1)^2 A \ge 0,$$

where

$$A = -20a^3+60a^2+15a-46.$$

The inequality (*) is true because

$$A \ge -20a^3+60a^2+15a-55 = (a-1)(-20a^2+40a+55)$$
$$\ge (a-1)(-20a^2+40a+25) = 5(a-1)(2a+1)(5-2a) \ge 0.$$

From $5 = ad+2(a+d) \ge ad+4\sqrt{ad}$ and $5 = ad+2(a+d) \le \frac{(a+d)^2}{4}+2(a+d)$, we get $ad \le 1$ and $a+d \ge 2$. So, to prove (**), it suffices to prove the homogeneous inequality

$$3(a^2-d^2)-5(ad)^{\frac{2}{3}}\left(a^{\frac{2}{3}}-d^{\frac{2}{3}}\right) \ge \frac{4(a-d)(a+d)}{3},$$

which is equivalent to

$$a^2-d^2-3(ad)^{\frac{2}{3}}\left(a^{\frac{2}{3}}-d^{\frac{2}{3}}\right) \ge 0, \quad \left(a^{\frac{2}{3}}-d^{\frac{2}{3}}\right)^3 \ge 0.$$

The proof is finished. If $k=\frac{5}{3}$, then the equality occurs for $a=b=c=d=1$.