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SP.602 Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = n$. Prove that:

$$\sum_{i=1}^{n} \sqrt{\frac{n-a_i}{n-1+a_i}} \geq \sqrt{n(n-1)}$$

Proposed by Vasile Cîrtoaje – Romania

Solution 1 by proposer

By Holder's inequality, we have

$$\left(\sum_{i=1}^{n} \sqrt{\frac{n-a_i}{n-1+a_i}}\right)^2 \sum_{i=1}^{n} (n-a_i)^2(n-1+a_i) \geq \left(\sum_{i=1}^{n} (n-a_i)\right)^3 = n^3(n-1)^3.$$

So, it suffices to show that

$$n^2(n-1)^2 \geq \sum_{i=1}^{n} (n-a_i)^2(a_i+n-1),$$

which is equivalent to $E \geq 0$, where

$$E = -\sum_{i=1}^{n} a_i^3 + (n+1)\sum_{i=1}^{n} a_i^2 - n^2.$$

For $n = 2$, we have $E = 0$, and for $n = 3$, the inequality $E \geq 0$ reduces to the well-known inequality

$$(a_1 + a_2 + a_3)(a_1a_2 + a_2a_3 + a_3a_1) \geq 9a_1a_2a_3.$$

For $n \geq 4$, assume $a_1 = \max\{a_1, a_2, \dots, a_n\}$ and consider two cases.

Case 1: $a_1 \leq n - 1$. We have

$$E = \sum_{i=1}^{n} [-a_i^3 + (n+1)a_i^2 - n] = \sum_{i=1}^{n} [-a_i^3 + (n+1)a_i^2 - (2n-1)a_i + n - 1]$$

$$= \sum_{i=1}^{n} (1-a_i)^2(n-1-a_i) \geq 0.$$

Case 2: $a_1 > n - 1$. From $a_1 + a_2 + \cdots + a_n = n$, it follows that $a_1 \in (n-1, n]$ and $a_2, \dots, a_n < 1$. So,

$$E \geq -a_1^3 + (n+1)a_1^2 - n^2 + a_2^2(1-a_2) + \cdots + a_n^2(1-a_n)$$

$$\geq -a_1^3 + (n+1)a_1^2 - n^2 = (n-a_1)(a_1^2 - a_1 - n).$$

Since $a_1^2 - a_1 - n > (n-1)^2 - (n-1) - n - n(n-4) + 2 > 0$, we have $E \geq 0$.

The proof is completed. The equality occurs for $a_1 = a_2 = \cdots = a_n$, and also for $a_1 = n$ and $a_2 = \cdots = a_n = 0$ (or any cyclic permutation).

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Write the inequality as

$$\sum_{i=1}^{n} f(a_i) \geq nf(s), \qquad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

Where

$$f(x) = \sqrt{\frac{n-x}{n-1+x}}, \quad x \in [0, n].$$

For $x \in [0,1]$, we have

$$f''(x) = \frac{(2n-1)(2n+1-4x)}{4\sqrt{(n-x)^3(n-1+x)^5}} \geq 0,$$

Then, f is convex on $[0, s]$. By the LHCF − Theorem [1], it suffice to prove that

$$f(x) + (n-1)f(y) \geq nf(1)$$

for $x \geq 1 \geq y \geq 0$ such that $x + (n-1)y = n$. The last inequality can be rewritten as

$$\sqrt{\frac{n-x}{n-1+x}} + (n-1)\sqrt{\frac{n-y}{n-1+y}} \geq \sqrt{n(n-1)}$$

$$\Leftrightarrow \sqrt{\frac{(n-1)y}{2n-1-(n-1)y}} + (n-1)\sqrt{\frac{n-y}{n-1+y}} \geq \sqrt{n(n-1)}.$$

Let $t = \sqrt{\frac{ny}{2n-1-(n-1)y}} \in [0,1]$, then $y = \frac{(2n-1)t^2}{n+(n-1)t^2}$.

The last inequality is equivalent to

$$\sqrt{\frac{(n-1)[n^2+(n^2-3n+1)t^2]}{n-1+nt^2}} \geq n - t$$

By squaring, we get the obvious inequality

$$nt(1-t)^2[2(n-1)-t] \geq 0,$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and for $a_1 = n, a_2 = \cdots = a_n = 0$, or any cyclic permutation.

[1] V. Cîrtoaje, *Mathematical Inequalities − Volume 4: Extensions and Refinements of Jensen's Inequality*, LAP LAMBERT Academic Publishing / Editura Universitatii Petrol − Gaze din Ploiesti, 2021, p. 4.