ROMANIAN MATHEMATICAL MAGAZINE

SP.601 What is the largest positive value of the power k such that

$$a^k + b^k + c^k \leq 3$$

for all nonnegative real numbers a, b, c, d with $a \leq b \leq c \leq d$ and $ab + bc + cd + da = 4$?

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Solution by proposer

For $a = 0$ and $b = c = d = \sqrt{2}$, the constraints are satisfied and the inequality becomes

$$2 \cdot 2^{\frac{k}{2}} \leq 3,$$

that is

$$k \leq m = \frac{\ln 9}{\ln 2} - 2 \approx 1.17, \quad 2^{\frac{m}{2}} = \frac{3}{2}.$$

To show that m is the largest positive k, we need to prove the required inequality for $k = m$. Let's write it in the homogeneous form

$$\left(\frac{ab + bc + cd + da}{4}\right)^{\frac{m}{2}} \geq \frac{a^m + b^m + c^m}{3}.$$

Since $d \geq c$, it suffices to prove the inequality for $d = c$. For fixed b and c, we need to prove that $f(a) \geq 0$, where

$$f(a) = \left(\frac{ab + bc + ca + c^2}{4}\right)^{\frac{m}{2}} - \frac{a^m + b^m + c^m}{3}, \quad a \in [0, b],$$

with

$$\frac{1}{m} f'(a) = \frac{b + c}{8}\left(\frac{4}{ab + bc + ca + c^2}\right)^{1-\frac{m}{2}} - \frac{1}{3} a^{m-1}.$$

Since $f'(a)$ is decreasing, $f(a)$ is concave. So, it suffices to prove that $f(a) \geq 0$ for $a = 0$ and $a = b$.

Case 1: $a = 0$. We need to show that

$$\left(\frac{bc + c^2}{4}\right)^{\frac{m}{2}} \geq \frac{b^m + c^m}{3}.$$

Due to homogeneity, we may set $c = 1$. So, we need to prove that

$$\left(\frac{b+1}{4}\right)^{\frac{m}{2}} \geq \frac{b^m+1}{3}$$

for $0 \leq b \leq 1$. Since $b^m \leq b$, it suffices to show that

$$\left(\frac{b+1}{4}\right)^{\frac{m}{2}} \geq \frac{b+1}{3},$$

which is true if

$$3 \geq 2^m(b+1)^{1-\frac{m}{2}}.$$

We have

$$3 - 2^m(b+1)^{1-\frac{m}{2}} \geq 3 - 2^m \cdot 2^{1-\frac{m}{2}} = 3 - 2 \cdot 2^{\frac{m}{2}} = 3 - 2 \cdot \frac{3}{2} = 0.$$

***Case 2:* $a = b$. We need to show that**

$$\left(\frac{b+c}{2}\right)^m \geq \frac{2b^m + c^m}{3}.$$

Due to homogeneity, we may set $c = 1$. So, we need to prove that

$$\left(\frac{b+1}{2}\right)^m \geq \frac{2b^m+1}{3}$$

for $0 \leq b \leq 1$. Since $b^m \leq b$, it suffices to show that

$$\left(\frac{b+1}{2}\right)^m \geq \frac{2b+1}{3}.$$

By Bernoulli's inequality, we have

$$\left(\frac{b+1}{2}\right)^m = \left(1 - \frac{1-b}{2}\right)^m \geq 1 - \frac{m}{2}(1-b) \geq 1 - \frac{2}{3}(1-b) = \frac{2b+1}{3}.$$

The proof is completed. If $k = m$, then the equality occurs for $a = b = c = d = 1$, as well as for $a = 0$ and $b = c = d = \sqrt{2}$.