Some Trigonometrical Inequalities in a Triangle

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Abstract

We prove a sharp half-angle estimate in an arbitrary triangle,

$$\max\left\{\frac{2-\sin(A/2)}{\cos(A/2)}, \frac{2-\sin(B/2)}{\cos(B/2)}, \frac{2-\sin(C/2)}{\cos(C/2)}\right\} \le \frac{s}{3r},$$

where s and r denote the semiperimeter and the inradius. Several corollaries involving the sides, exradii, inradius, and circumradius are then obtained by multiplying this sharp estimate by standard positive geometric factors. Ambiguous cyclic statements are written below in their corrected sharp form.

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1 Introduction

Triangle inequalities involving s, r, R and the exradii r_a, r_b, r_c form a classical part of Euclidean geometry; see, for instance, [1, 2]. The purpose of this paper is to give a compact proof of a sharp estimate for

$$\frac{2-\sin(A/2)}{\cos(A/2)}$$

and to derive a family of consequences that are useful in olympiad-style and journal problem settings such as [3, 4].

Throughout the paper ABC is a triangle with side lengths a, b, c, semiperimeter $s = (a+b+c)/2$, area Δ, inradius r, circumradius R, and exradii r_a, r_b, r_c. We use the cyclic notation

$$\sum_{\text{cyc}} f(A,B,C) = f(A,B,C) + f(B,C,A) + f(C,A,B).$$

For brevity put

$$p_A = 2 - \sin\frac{A}{2}, \qquad c_A = \cos\frac{A}{2}, \qquad X_A = \frac{p_A}{c_A},$$

and define p_B, c_B, X_B and p_C, c_C, X_C cyclically.

Remark 1.1. *If a displayed symbol $\sum$ is read literally in the first product group of the scanned manuscript, the first two inequalities are false for the equilateral triangle. The sharp statements are the individual cyclic-term estimates in Corollary 4.1; true cyclic-sum consequences are also recorded there.*

2 Auxiliary formulae

Lemma 2.1. *The following standard identities hold:*

$$c_A c_B c_C = \frac{s}{4R}, \tag{1}$$

$$\sum_{\text{cyc}} \sin^2 \frac{A}{2} = \frac{2R-r}{2R}, \qquad \sum_{\text{cyc}} c_A^2 = \frac{4R+r}{2R}, \tag{2}$$

$$\sum_{\text{cyc}} c_A^4 = \frac{(4R+r)^2 - s^2}{8R^2}, \qquad \sum_{\text{cyc}} c_A^6 = \frac{(4R+r)^3 - 3s^2(2R+r)}{32R^3}, \tag{3}$$

$$\sum_{\text{cyc}} r_a = 4R + r, \qquad \sum_{\text{cyc}} r_b r_c = s^2, \qquad \sum_{\text{cyc}} \frac{a}{r_a} = \frac{2(4R+r)}{s}. \tag{4}$$

Proof. We use the half-angle formulae

$$\sin^2 \frac{A}{2} = \frac{(s-b)(s-c)}{bc}, \qquad \cos^2 \frac{A}{2} = \frac{s(s-a)}{bc},$$

together with Heron's formula $\Delta^2 = s(s-a)(s-b)(s-c)$, $\Delta = rs$, and $abc = 4R\Delta$. For example,

$$c_A c_B c_C = \sqrt{\frac{s(s-a)}{bc}\frac{s(s-b)}{ca}\frac{s(s-c)}{ab}} = \frac{s\Delta}{abc} = \frac{s}{4R}.$$

The two identities in (2) follow by summing the half-angle formulae.

Let $u_A = c_A^2$, $u_B = c_B^2$, $u_C = c_C^2$. A direct calculation from the same formulae gives

$$\begin{aligned} u_A + u_B + u_C &= \frac{4R+r}{2R}, \\ u_A u_B + u_B u_C + u_C u_A &= \frac{(4R+r)^2 + s^2}{16R^2}, \\ u_A u_B u_C &= \frac{s^2}{16R^2}. \end{aligned}$$

Newton's identities therefore yield the fourth- and sixth-power formulae in (3).

Finally, set $x = s-a$, $y = s-b$, $z = s-c$. Then $s = x+y+z$, $a = y+z$, $b = z+x$, $c = x+y$, and $\Delta^2 = sxyz$. Since $r_a = \Delta/x$ and cyclically,

$$\sum_{\text{cyc}} r_a = \Delta\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right), \qquad \sum_{\text{cyc}} r_b r_c = \Delta^2\left(\frac{1}{yz} + \frac{1}{zx} + \frac{1}{xy}\right) = s^2.$$

The identity

$$(x+y)(y+z)(z+x) + xyz = s(xy + yz + zx)$$

gives $\sum_{\text{cyc}} r_a = 4R + r$. Also,

$$\sum_{\text{cyc}} \frac{a}{r_a} = \frac{x(y+z) + y(z+x) + z(x+y)}{\Delta} = \frac{2(xy + yz + zx)}{\Delta} = \frac{2(4R+r)}{s}.$$

□

3 The main estimate

Theorem 3.1. *In every triangle ABC,*

$$\max\{X_A, X_B, X_C\} \le \frac{s}{3r}.$$

Equivalently,

$$\max\left\{\frac{2-\sin\frac{A}{2}}{\cos\frac{A}{2}}, \frac{2-\sin\frac{B}{2}}{\cos\frac{B}{2}}, \frac{2-\sin\frac{C}{2}}{\cos\frac{C}{2}}\right\} \le \frac{s}{3r}.$$

Equality holds if and only if ABC is equilateral.

Proof. It is enough to prove $X_A \le s/(3r)$. Let

$$x = \frac{A}{2}, \qquad y = \frac{B}{2}, \qquad z = \frac{C}{2}.$$

Then $x + y + z = \pi/2$. Since $\tan(A/2) = r/(s-a)$,

$$\frac{s}{r} = \cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot x + \cot y + \cot z.$$

The function $t \mapsto \cot t$ is convex on $(0, \pi/2)$, hence

$$\cot y + \cot z \ge 2\cot\frac{y+z}{2} = 2\cot\left(\frac{\pi}{4} - \frac{x}{2}\right).$$

Set $u = \tan(x/2)$. Since $0 < u < 1$, a direct simplification gives

$$\begin{aligned}\cot x + 2\cot\left(\frac{\pi}{4} - \frac{x}{2}\right) - 3\frac{2-\sin x}{\cos x}& \\ = \frac{(u^2-4u+1)^2}{2u(1-u)(1+u)} &\ge 0.\end{aligned}$$

Thus

$$\frac{s}{r} \ge 3\frac{2-\sin(A/2)}{\cos(A/2)},$$

and so $X_A \le s/(3r)$. The equality conditions are $y = z$ and $u^2 - 4u + 1 = 0$. The admissible root is $u = 2 - \sqrt{3}$, which gives $x = \pi/6$ and hence $A = B = C = 60°$. □

In the sequel put

$$M = \frac{s}{3r}.$$

Theorem 3.1 says that $0 < X_A, X_B, X_C \le M$.

4 Consequences

Corollary 4.1 (product estimates). *For each cyclic choice of* (A, B, C),

$$p_A c_B c_C \le \frac{s^2}{12Rr}, \tag{5}$$

$$p_A p_B c_C \le \frac{s^3}{36Rr^2}, \tag{6}$$

$$p_A p_B p_C \le \frac{s^4}{108Rr^3}. \tag{7}$$

Consequently,

$$\sum_{\text{cyc}} p_A c_B c_C \le \frac{s^2}{4Rr}, \tag{8}$$

$$\sum_{\text{cyc}} p_A p_B c_C \le \frac{s^3}{12Rr^2}. \tag{9}$$

Proof. Using $p_A = X_A c_A$ and (1),

$$p_A c_B c_C = X_A c_A c_B c_C \le M \frac{s}{4R} = \frac{s^2}{12Rr}.$$

Similarly,

$$p_A p_B c_C = X_A X_B c_A c_B c_C \le M^2 \frac{s}{4R} = \frac{s^3}{36Rr^2},$$

and

$$p_A p_B p_C = X_A X_B X_C c_A c_B c_C \le M^3 \frac{s}{4R} = \frac{s^4}{108Rr^3}.$$

The two displayed cyclic-sum estimates follow by summing the first two individual estimates. $\square$

Corollary 4.2 (upper cyclic estimates). *In every triangle,*

$$\sum_{\text{cyc}} r_a X_A \le \frac{s(4R+r)}{3r}, \tag{10}$$

$$\sum_{\text{cyc}} X_A \sin^2 \frac{B}{2} \le \frac{s(2R-r)}{6Rr}, \qquad \sum_{\text{cyc}} X_A \sin^2 \frac{C}{2} \le \frac{s(2R-r)}{6Rr}, \tag{11}$$

$$\sum_{\text{cyc}} p_A c_A \le \frac{s(4R+r)}{6Rr}, \tag{12}$$

$$\sum_{\text{cyc}} p_A c_A^3 \le \frac{s\big((4R+r)^2 - s^2\big)}{24R^2 r}, \tag{13}$$

$$\sum_{\text{cyc}} p_A c_A^5 \le \frac{s\big((4R+r)^3 - 3s^2(2R+r)\big)}{96R^3 r}. \tag{14}$$

Proof. From $X_A \le M$ and Lemma 2.1,

$$\sum_{\text{cyc}} r_a X_A \le M \sum_{\text{cyc}} r_a = \frac{s(4R+r)}{3r}.$$

Furthermore,

$$\sum_{\text{cyc}} X_A \sin^2 \frac{B}{2} \le M \sum_{\text{cyc}} \sin^2 \frac{B}{2} = \frac{s}{3r} \cdot \frac{2R-r}{2R} = \frac{s(2R-r)}{6Rr},$$

and the same proof gives the version with C in place of B. Finally,

$$\sum_{\text{cyc}} p_A c_A = \sum_{\text{cyc}} X_A c_A^2, \quad \sum_{\text{cyc}} p_A c_A^3 = \sum_{\text{cyc}} X_A c_A^4, \quad \sum_{\text{cyc}} p_A c_A^5 = \sum_{\text{cyc}} X_A c_A^6.$$

Applying $X_A \le M$ and the identities (2)–(3) gives (12), (13), and (14). □

Corollary 4.3 (reciprocal estimates)**.** *In every triangle,*

$$\sum_{\text{cyc}} \frac{c_A}{p_A} \ge \frac{9r}{s}, \tag{15}$$

$$\sum_{\text{cyc}} \frac{c_C}{p_A p_B} \ge \frac{18r^2(4R+r)}{s^3} \ge \frac{36Rr^2}{s^3}. \tag{16}$$

Proof. Since $X_A \le M$, we have $1/X_A \ge 1/M = 3r/s$. Hence

$$\sum_{\text{cyc}} \frac{c_A}{p_A} = \sum_{\text{cyc}} \frac{1}{X_A} \ge 3 \cdot \frac{3r}{s} = \frac{9r}{s}.$$

Also,

$$\frac{c_C}{p_A p_B} = \frac{c_C}{X_A X_B c_A c_B} \ge \frac{1}{M^2} \frac{c_C}{c_A c_B}.$$

By (1) and (2),

$$\sum_{\text{cyc}} \frac{c_C}{c_A c_B} = \frac{\sum_{\text{cyc}} c_C^2}{c_A c_B c_C} = \frac{\frac{4R+r}{2R}}{\frac{s}{4R}} = \frac{2(4R+r)}{s}.$$

Therefore

$$\sum_{\text{cyc}} \frac{c_C}{p_A p_B} \ge \frac{9r^2}{s^2} \cdot \frac{2(4R+r)}{s} = \frac{18r^2(4R+r)}{s^3}.$$

The last inequality follows from $4R + r \ge 2R$. □

Corollary 4.4 (more reciprocal estimates)**.** *In every triangle,*

$$\sum_{\text{cyc}} \frac{r_a c_A}{p_A} \geq \frac{3r(4R+r)}{s}, \tag{17}$$

$$\sum_{\text{cyc}} \frac{r_b r_c c_A}{p_A} \geq 3sr, \tag{18}$$

$$\sum_{\text{cyc}} \frac{c_A^3}{p_A} \geq \frac{3r(4R+r)}{2sR}, \tag{19}$$

$$\sum_{\text{cyc}} \frac{c_A^5}{p_A} \geq \frac{3r\big((4R+r)^2 - s^2\big)}{8sR^2}, \tag{20}$$

$$\sum_{\text{cyc}} \frac{c_A^7}{p_A} \geq \frac{3r\big((4R+r)^3 - 3s^2(2R+r)\big)}{32sR^3}. \tag{21}$$

Proof. Each estimate follows from $1/X_A \geq 3r/s$. Indeed,

$$\sum_{\text{cyc}} \frac{r_a c_A}{p_A} = \sum_{\text{cyc}} \frac{r_a}{X_A} \geq \frac{3r}{s} \sum_{\text{cyc}} r_a = \frac{3r(4R+r)}{s},$$

and

$$\sum_{\text{cyc}} \frac{r_b r_c c_A}{p_A} = \sum_{\text{cyc}} \frac{r_b r_c}{X_A} \geq \frac{3r}{s} \sum_{\text{cyc}} r_b r_c = 3sr.$$

The remaining estimates are

$$\sum_{\text{cyc}} \frac{c_A^3}{p_A} = \sum_{\text{cyc}} \frac{c_A^2}{X_A}, \qquad \sum_{\text{cyc}} \frac{c_A^5}{p_A} = \sum_{\text{cyc}} \frac{c_A^4}{X_A}, \qquad \sum_{\text{cyc}} \frac{c_A^7}{p_A} = \sum_{\text{cyc}} \frac{c_A^6}{X_A},$$

combined with (2)–(3). □

Corollary 4.5 (side-weighted estimates)**.** *In every triangle,*

$$\sum_{\text{cyc}} \frac{a p_A}{c_A} \leq \frac{2s^2}{3r}, \tag{22}$$

$$\sum_{\text{cyc}} \frac{a c_A}{p_A} \geq 6r, \tag{23}$$

$$\sum_{\text{cyc}} \frac{a p_A}{r_a c_A} \leq \frac{2(4R+r)}{3r}, \tag{24}$$

$$\sum_{\text{cyc}} \frac{a c_A}{r_a p_A} \geq \frac{6r(4R+r)}{s^2}. \tag{25}$$

Proof. Since $a+b+c = 2s$,

$$\sum_{\text{cyc}} \frac{a p_A}{c_A} = \sum_{\text{cyc}} a X_A \leq M \sum_{\text{cyc}} a = \frac{s}{3r} \cdot 2s = \frac{2s^2}{3r},$$

and

$$\sum_{\text{cyc}} \frac{ac_A}{p_A} = \sum_{\text{cyc}} \frac{a}{X_A} \geq \frac{3r}{s} \sum_{\text{cyc}} a = 6r.$$

Using (4),

$$\sum_{\text{cyc}} \frac{ap_A}{r_a c_A} = \sum_{\text{cyc}} \frac{a}{r_a} X_A \leq M \sum_{\text{cyc}} \frac{a}{r_a} = \frac{s}{3r} \cdot \frac{2(4R+r)}{s} = \frac{2(4R+r)}{3r},$$

and

$$\sum_{\text{cyc}} \frac{ac_A}{r_a p_A} = \sum_{\text{cyc}} \frac{a/r_a}{X_A} \geq \frac{3r}{s} \sum_{\text{cyc}} \frac{a}{r_a} = \frac{6r(4R+r)}{s^2}.$$

□

5 Equality cases

The main theorem is sharp, with equality only in the equilateral triangle. All estimates above that are obtained by replacing every X_A by the common upper bound M are sharp when the triangle is equilateral. In Corollary 4.3, the strengthened first lower bound in (16) is sharp in this sense; the weaker displayed consequence $36Rr^2/s^3$ is included because it is a simpler bound.

References

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