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S.3327. *Proposed by Dumitru Bătineţu-Giurgiu, and Claudia Nănuţi, Romania.* In any triangle ABC with area F and medians m_a, m_b, m_c, the following inequality holds

$$m_a^4 + m_b^4 + m_c^4 + 3 \geq 6\sqrt{3}\,F$$

Solution by José Luis Díaz Barrero, Barcelona, Spain. By the AM-GM Inequality, for any real number t, we have $t^2 + 1 \geq 2t$. Applying this to each median squared (m_a^2, m_b^2, m_c^2) we have

$$\begin{aligned} m_a^4 + 1 &\geq 2m_a^2 \\ m_b^4 + 1 &\geq 2m_b^2 \\ m_c^4 + 1 &\geq 2m_c^2 \end{aligned}$$

Summing these three inequalities yields

$$m_a^4 + m_b^4 + m_c^4 + 3 \geq 2(m_a^2 + m_b^2 + m_c^2)$$

By Apollonius's Theorem, the sum of the squares of the medians for a triangle with sides a, b, c is given by

$$m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2)$$

Substituting this into the right-hand side yields

$$2(m_a^2 + m_b^2 + m_c^2) = 2 \cdot \frac{3}{4}(a^2 + b^2 + c^2) = \frac{3}{2}(a^2 + b^2 + c^2)$$

Weitzenböck's Inequality, for any triangle with sides a, b, c and area F states

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}\,F$$

Substituting this lower bound gives

$$\frac{3}{2}(a^2 + b^2 + c^2) \geq \frac{3}{2}\left(4\sqrt{3}\,F\right) = 6\sqrt{3}\,F$$

Finally, combining all previous steps yields

$$m_a^4 + m_b^4 + m_c^4 + 3 \geq 2(m_a^2 + m_b^2 + m_c^2) \geq 6\sqrt{3}\,F$$

Equality holds if and only if the triangle is equilateral ($a = b = c = 2\sqrt{3}/3$) and $m_a = m_b = m_c = 1$.