José Luis **Díaz-Barrero**
Barcelona Mathematical Circle (BMC)
jose.luis.diaz@upc.edu

S.3326. *Proposed by Dumitru Bătineţu-Giurgiu, and Claudia Nănuţi, Romania.* In any triangle ABC with area F and altitudes h_a, h_b, h_c, the following inequality holds

$$(a^3+b^3+c^3)\left(\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\right) \geq 24F$$

Solution by José Luis Díaz Barrero, Barcelona, Spain. The area F of triangle ABC can be expressed using each side and its corresponding altitude as

$$F = \frac{1}{2}ah_a = \frac{1}{2}bh_b = \frac{1}{2}ch_c$$

Solving for the reciprocals of the altitudes yields

$$\frac{1}{h_a} = \frac{a}{2F}, \quad \frac{1}{h_b} = \frac{b}{2F}, \quad \frac{1}{h_c} = \frac{c}{2F}$$

Summing these reciprocals, we obtain

$$\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c} = \frac{a+b+c}{2F}$$

Substituting this into the left-hand side gives

$$(a^3+b^3+c^3)\left(\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\right) = \frac{(a^3+b^3+c^3)(a+b+c)}{2F}$$

By the Power Mean Inequality, we have

$$\frac{a^3+b^3+c^3}{3} \geq \left(\frac{a+b+c}{3}\right)^3 \Rightarrow a^3+b^3+c^3 \geq \frac{(a+b+c)^3}{9}$$

Multiplying both sides by $(a+b+c)$ yields

$$(a^3+b^3+c^3)(a+b+c) \geq \frac{(a+b+c)^4}{9}$$

The Isoperimetric Inequality for triangles with side lengths a, b, c and area F states

$$F \leq \frac{\sqrt{3}}{36}(a+b+c)^2 \implies (a+b+c)^2 \geq 12\sqrt{3}\,F$$

Squaring both sides gives:

$$(a+b+c)^4 \geq (12\sqrt{3}\,F)^2 = 432F^2$$

Substituting $(a+b+c)^4 \geq 432F^2$ into the inequality from the preceding, yields

$$(a^3+b^3+c^3)(a+b+c) \geq \frac{432F^2}{9} = 48F^2$$

Finally, substituting this into the previous expression yields

$$(a^3+b^3+c^3)\left(\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\right) = \frac{(a^3+b^3+c^3)(a+b+c)}{2F} \geq \frac{48F^2}{2F} = 24F$$

Equality holds if and only if $a=b=c$. That is, when the triangle is equilateral.