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S.3325. *Proposed by Dumitru Bătineţu-Giurgiu, and Neculai Stanciu, Romania.* If $n \geq 3, a_k > 0, \forall k = \overline{1,n}, s_n = \sum_{k=1}^n a_k$, and $x, y > 0$ such that $xs_n > y \cdot \max_{1 \leq k \leq n} a_k$, then prove that

$$\left(\sum_{k=1}^n \frac{a_k \sqrt[3]{a_k}}{xs_n - ya_k}\right) \cdot \left(\sum_{k=1}^n \frac{\sqrt[3]{a_k^2}}{xs_n - ya_k}\right) \geq \frac{n^2}{(nx-y)^2}$$

Solution by José Luis Díaz Barrero, Barcelona, Spain. Note that $a_k \sqrt[3]{a_k} = a_k^{4/3}$ and $\sqrt[3]{a_k^2} = a_k^{2/3}$. On account of CBS, we have

$$\left(\sum_{k=1}^n \frac{a_k^{4/3}}{xs_n - ya_k}\right) \cdot \left(\sum_{k=1}^n \frac{a_k^{2/3}}{xs_n - ya_k}\right) \geq \left(\sum_{k=1}^n \sqrt{\frac{a_k^{4/3}}{xs_n - ya_k}} \cdot \sqrt{\frac{a_k^{2/3}}{xs_n - ya_k}}\right)^2$$

Simplifying the terms inside the square root on the right-hand side yields

$$\sqrt{\frac{a_k^{4/3}}{xs_n - ya_k}} \cdot \sqrt{\frac{a_k^{2/3}}{xs_n - ya_k}} = \frac{\sqrt{a_k^2}}{xs_n - ya_k} = \frac{a_k}{xs_n - ya_k}$$

Thus, we have

$$\left(\sum_{k=1}^n \frac{a_k \sqrt[3]{a_k}}{xs_n - ya_k}\right) \cdot \left(\sum_{k=1}^n \frac{\sqrt[3]{a_k^2}}{xs_n - ya_k}\right) \geq \left(\sum_{k=1}^n \frac{a_k}{xs_n - ya_k}\right)^2$$

To evaluate $\displaystyle\sum_{k=1}^n \frac{a_k}{xs_n - ya_k}$, rewrite each term as $\dfrac{1}{\frac{xs_n}{a_k} - y}$ to obtain

$$\sum_{k=1}^n \frac{a_k}{xs_n - ya_k} = \sum_{k=1}^n \frac{1}{\frac{xs_n}{a_k} - y}$$

By Cauchy-Schwarz (in the form $\displaystyle\sum \frac{1}{t_k} \geq \frac{n^2}{\sum t_k}$), we have

$$\sum_{k=1}^n \frac{1}{\frac{xs_n}{a_k} - y} \geq \frac{n^2}{\sum_{k=1}^n \left(\frac{xs_n}{a_k} - y\right)} = \frac{n^2}{xs_n \left(\sum_{k=1}^n \frac{1}{a_k}\right) - ny}$$

By the AM-HM inequality, we know that $\sum_{k=1}^n \frac{1}{a_k} \geq \frac{n^2}{s_n}$. Substituting this into the denominator gives

$$xs_n \left(\sum_{k=1}^n \frac{1}{a_k}\right) - ny \geq xs_n \left(\frac{n^2}{s_n}\right) - ny = n^2 x - ny = n(nx-y)$$

Therefore,

$$\sum_{k=1}^{n} \frac{a_k}{xs_n - ya_k} \geq \frac{n^2}{n(nx-y)} = \frac{n}{nx-y}$$

Squaring both sides yields

$$\left(\sum_{k=1}^{n} \frac{a_k}{xs_n - ya_k}\right)^2 \geq \frac{n^2}{(nx-y)^2}$$

Combining this with the preceding we complete the proof

$$\left(\sum_{k=1}^{n} \frac{a_k \sqrt[3]{a_k}}{xs_n - ya_k}\right) \cdot \left(\sum_{k=1}^{n} \frac{\sqrt[3]{a_k^2}}{xs_n - ya_k}\right) \geq \left(\sum_{k=1}^{n} \frac{a_k}{xs_n - ya_k}\right)^2 \geq \frac{n^2}{(nx-y)^2}$$

Equality holds if and only if $a_1 = a_2 = \cdots = a_n$.