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S.3324. *Proposed by Dumitru Bătineţu-Giurgiu, and Mihály Bencze, Romania.* If $n \in \mathbb{N}, n \geq 3, a_k \in \mathbb{R}_+^* = (0, \infty), \forall k = \overline{1, n}, s_n = \sum_{k=1}^n a_k$, and $x, y \in \mathbb{R}_+^*$ such that $xs_n > y \cdot \max_{1 \leq k \leq n} a_k$, then prove that

$$\left(\sum_{k=1}^n \frac{a_k^3}{xs_n - ya_k} \right) \cdot \left(\sum_{k=1}^n \frac{a_k}{xs_n - ya_k} \right) \geq \frac{s_n^2}{(nx - y)^2}$$

Solution by José Luis Díaz Barrero, Barcelona, Spain. By the Cauchy-Schwarz inequality, we have

$$\left(\sum_{k=1}^n \frac{a_k^3}{xs_n - ya_k} \right) \cdot \left(\sum_{k=1}^n \frac{a_k}{xs_n - ya_k} \right) \geq \left(\sum_{k=1}^n \sqrt{\frac{a_k^3}{xs_n - ya_k}} \cdot \sqrt{\frac{a_k}{xs_n - ya_k}} \right)^2$$

Simplifying the terms inside the square root on the right-hand side, we get

$$\sqrt{\frac{a_k^3}{xs_n - ya_k}} \cdot \sqrt{\frac{a_k}{xs_n - ya_k}} = \frac{\sqrt{a_k^4}}{xs_n - ya_k} = \frac{a_k^2}{xs_n - ya_k}$$

Thus,

$$\left(\sum_{k=1}^n \frac{a_k^3}{xs_n - ya_k} \right) \cdot \left(\sum_{k=1}^n \frac{a_k}{xs_n - ya_k} \right) \geq \left(\sum_{k=1}^n \frac{a_k^2}{xs_n - ya_k} \right)^2$$

Using again Cauchy's inequality, we have

$$\sum_{k=1}^n \frac{a_k^2}{xs_n - ya_k} \geq \frac{\left(\sum_{k=1}^n a_k\right)^2}{\sum_{k=1}^n (xs_n - ya_k)}$$

Next, we calculate the sum in the denominator

$$\sum_{k=1}^n (xs_n - ya_k) = \sum_{k=1}^n xs_n - y \sum_{k=1}^n a_k = nxs_n - ys_n = s_n(nx - y)$$

Substituting this back into the preceding gives

$$\sum_{k=1}^n \frac{a_k^2}{xs_n - ya_k} \geq \frac{s_n^2}{s_n(nx - y)} = \frac{s_n}{nx - y}$$

Squaring both sides yields

$$\left(\sum_{k=1}^n \frac{a_k^2}{xs_n - ya_k} \right)^2 \geq \frac{s_n^2}{(nx - y)^2}$$

Combining this with the preceding we complete the proof

$$\left(\sum_{k=1}^{n} \frac{a_k^3}{xs_n - ya_k}\right) \cdot \left(\sum_{k=1}^{n} \frac{a_k}{xs_n - ya_k}\right) \geq \left(\sum_{k=1}^{n} \frac{a_k^2}{xs_n - ya_k}\right)^2 \geq \frac{s_n^2}{(nx-y)^2}$$

Equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

Note. A natural questions arises. Where is the condition $xs_n > y \cdot \max_{1 \leq k \leq n} a_k$ used? The condition is used to ensure that all denominators in the inequality are strictly positive $xs_n - ya_k > 0 \quad \text{for all } k = 1, 2, \ldots, n$.